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Riesz potential operator in continual variable exponents Herz spaces

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We find conditions on the variable parameters $p(x)$, $q(t)$ and $\alpha(t)$, defining the Herz space $H^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$, for the validity of Sobolev type theorem for the Riesz potential operator to be bounded within the frameworks of such variable exponents Herz spaces. We deal with a "continual" version of Herz spaces (which coincides with the "discrete" one when q is constant).

1 Introduction

The classical versions of Herz spaces $K^{\alpha}_{p,q}(\mathbb{R}^n)$ and $\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$, introduced in [13], known under the names of *nonhomogeneous* and *homogeneous* Herz spaces, are defined by the norms

$$\|f\|_{K^{\alpha}_{p,q}} := \|f\|_{L^p(B(0,1))} + \left\{\sum_{k\in\mathbb{N}} 2^{k\alpha q}\left(\int_{\mathsf{R}(2^{k-1},2^k)} |f(x)|^p\,dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}} \tag{1.1}$$

$$\|f\|_{\dot{K}^{\alpha}_{p,q}} := \left\{\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\left(\int_{\mathsf{R}(2^{k-1},2^k)} |f(x)|^p\,dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}, \tag{1.2}$$

respectively, where $\mathsf{R}(t,\tau)$ stands for the annulus $\mathsf{R}(t,\tau) := B(0,\tau)\backslash B(0,t)$. These spaces were studied in many papers, see for instance [7], [8], [12], [14], [15], [17], [18], [23] and references therein.

In the last two decades, under the influence of some applications revealed in [28], there was a vast boom of research of the so-called variable exponent spaces, and operators in them, where the parameters defining the space or the operator, may depend on the point x of the underlying space. For the time being, the theory of such variable exponent Lebesgue, Orlicz, Lorentz, and Sobolev function spaces is widely developed, we refer to the recent books [4], [6] and the surveying papers [5], [19], [22], [30]. For variable exponent Morrey-Campanato spaces we refer to the papers [2], [9]–[11], [20], [21], [26], [27].

Herz spaces with variables exponents have been recently introduced in [1], [14], [15]. In the last two papers the exponent p was variable, the remaining exponents α and q were kept constant. The most general results were obtained in [1], where the variability of α was allowed. The main results obtained, for instance in [1] concern the boundedness of sublinear operators (including the maximal function and Calderón-Zygmund singular operators) and a Spanne type result for the Riesz potential operator. The approach used in [1] allowed to cover the case where p and α are variable and depend on the point x of the underlying set, keeping the exponent q constant.

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Recently, in [31], [32] there was used another approach, based on the "continual" interpretation of Herz spaces via the norm

$$k^{\alpha}_{p,q}(f) := \|f\|_{L^p(B(0,\gamma\nu+\varepsilon))} + \left\{ \int_{\nu}^{\infty} t^{\alpha q} \left(\int_{\gamma t<|x|<\delta t} |f(x)|^p \, dx \right)^{\frac{q}{p}} \frac{dt}{t} \right\}^{\frac{1}{q}} \tag{1.3}$$

where $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, $\alpha \in \mathbb{R}$, $\varepsilon > 0$, $\nu \geqslant 0$ and $0 < \gamma < \delta < \infty$, the cases $\nu = 0$ and $\nu > 0$ corresponding to homogeneous and non-homogeneous Herz spaces (the first term in (1.3) should be omitted in the case $\nu = 0$). This norm is equivalent to the classical Herz norm given by discrete ℓ^q-norm, see [31] for more details. Based on this approach, in [31], [32] there were introduced the corresponding variable exponent Herz spaces where all the three main parameters p, q and α are variable, and sublinear operators of singular type were studied in such spaces (which include maximal functions and Calderón-Zygmund type singular operators). Within this approach $p(x)$ is defined on $\mathbb{R}^n$, while $q(t)$ and $\alpha(t)$ are functions on $\mathbb{R}_+$. A different generalization to include the case of variable $q(\cdot)$ was also given in [16].

In this paper we extend the approach of [31], [32] to the case of potential operators and prove the corresponding Sobolev type theorem for variable exponent Herz spaces. We write down the proof for the Riesz potential operator, but note that the proof is valid for a more general case of sublinear operators with potential type size condition, which are bounded on variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. When all the exponents are constants, we recover the result proved in [25], which states, shortly speaking, that if a sublinear operator satisfies a certain size condition and is bounded in the Lebesgue space is also bounded in the appropriate Herz space.

Notation:

$B(x, r)$ is the ball of radius r centered at the point x;
$\mathsf{R}(t, \tau) := B(0, \tau) \backslash B(0, t) = \{x \in \mathbb{R}^n : t < |x| < \tau\}$ is a spherical layer;
$\mathsf{R}_k := \mathsf{R}(2^{k-1}, 2^k)$;
$\chi_E(x)$ is the characteristic function of a set E;
$\mathbb{R}_{\nu+} := (\nu, \infty)$, where $\nu \geqslant 0$;
$\chi_{t,\tau}(x) = \chi_{\mathsf{R}(t,\tau)}(x)$;
dt/t denotes the Haar measure on $\mathbb{R}_+$;
$\mathbb{N}$ is the set of all natural numbers;
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$;
$\mathbb{Z}$ is the set of all integers;
$f \lesssim g$ for nonnegative f and g means that $f \leqslant Cg$, where C does not depend on variables involved in f and g.

2 Preliminaries

2.1 Spaces of variable integrability

We refer to the books [4], [6] and papers [24], [30], but recall some basics we need on *variable exponent Lebesgue spaces*. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $p(\cdot)$ be a real-valued measurable function on Ω with values in $[1, \infty)$. We suppose that

$$1 \leqslant p_- \leqslant p(x) \leqslant p_+ < \infty, \tag{2.1}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$, $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$. By $L^{p(\cdot)}(\Omega)$ we denote the space of measurable functions f on Ω such that

$$I_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} \, dx < \infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf \left\{ \eta > 0 : I_{p(\cdot)}\left(\frac{f}{\eta}\right) \leqslant 1 \right\},$$

this is a Banach function space. By $p'(x) = p(x)/(p(x) - 1)$, we denote the conjugate exponent.

In the sequel we use the well-known log-*condition*

$$|p(x)-p(y)|\leqslant\frac{A}{-\ln|x-y|},\quad |x-y|\leqslant\frac{1}{2},\quad x,y\in\Omega, \tag{2.2}$$

where $A=A(p)>0$ does not depend on x, y, and *the decay condition*: there exists a number $p_\infty\in(1,\infty)$, such that

$$|p(x)-p_\infty|\leqslant\frac{A}{\ln(e+|x|)}, \tag{2.3}$$

and also the decay condition

$$|p(x)-p_0|\leqslant\frac{A}{\ln|x|},\quad |x|\leqslant\frac{1}{2}, \tag{2.4}$$

in case of homogeneous Herz spaces.

With respect to classes of variable exponents used in this paper, we adopt the following notation:

(a) $\mathscr{P}^{\log}=\mathscr{P}^{\log}(\Omega)$ is the class of functions $p\in L^\infty(\Omega)$ satisfying the conditions (2.1) and (2.2);
(b) in the case Ω is unbounded, $\mathscr{P}_\infty(\Omega)$ and $\mathscr{P}_{0,\infty}(\Omega)$ are the subsets of exponents in $L^\infty(\Omega)$ with values in $[1,\infty)$ which satisfy the condition (2.3) and both the conditions (2.3) and (2.4), respectively; $\mathscr{P}^{\log}_\infty(\Omega)$ is the set of exponents $p\in\mathscr{P}_\infty(\Omega)$, satisfying also the condition (2.2);
(c) in the case $\Omega=\mathbb{R}_{\nu+}$, $\nu\geqslant 0$, we also use the class $\mathscr{M}_\infty(\mathbb{R}_{\nu+})$ of functions g such that $g(t)=const+g_0(t)$, where $g_0\in\mathscr{P}_\infty(\mathbb{R}_{\nu+})$.
(d) in the case $\Omega=\mathbb{R}_+$ (the case $\nu=0$), $\mathscr{M}_{0,\infty}(\mathbb{R}_+)$ is the class of functions $g\in\mathscr{M}_\infty(\mathbb{R}_+)$ which satisfy the decay condition also at the origin: that there exist a real numbers g_0 such that $|g(x)-g_0|\leqslant\frac{A}{|\ln x|}$, $0<x\leqslant\frac{1}{2}$. We also write $g_0=g(0)$, $g_\infty=g(\infty)$ in this case;
(e) $\mathscr{P}_{0,\infty}(\mathbb{R}_+)$ is the subclass of functions in $\mathscr{M}_{0,\infty}(\mathbb{R}_+)$ with values in $[1,\infty)$.

We will also work with the variable exponent Lebesgue space with the Haar measure dt/t on $\mathbb{R}_{\nu+}$, $\nu\geqslant 0$, which is introduced in the usual way:

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}_{\nu+};dt/t)}=\inf\left\{\eta>0:\int_\nu^\infty\left|\frac{f(t)}{\eta}\right|^{q(t)}\frac{dt}{t}\leqslant 1\right\}.$$

2.2 Technical lemmata

In this subsection we collect some technical lemmata that we will need.

Let

$$K\varphi(t)=\int_0^\infty\mathcal{K}\left(\tfrac{t}{\tau}\right)\varphi(\tau)\,\frac{d\tau}{\tau} \tag{2.5}$$

be an integral operator with the Haar measure $d\tau/\tau$ and the kernel homogeneous of order 0, known also as *Mellin convolution operator*. We refer to [33] for Mellin convolution operators in variable exponent Lebesgue spaces.

Lemma 2.1 ([31, Corollary 4.5] and [33]) *Let $q\in\mathscr{P}_{0,\infty}(\mathbb{R}_+)$ and $q(0)=q(\infty)$. The operator K is bounded in the space $L^{q(\cdot)}(\mathbb{R}_+;dt/t)$, if*

$$\int_0^\infty|\mathcal{K}(t)|^s\frac{dt}{t}<\infty\quad\text{for}\quad s=1\ \text{ and }\ s=s_0, \tag{2.6}$$

where $\frac{1}{s_0}=1-\frac{1}{q_-}+\frac{1}{q_+}$.

Lemma 2.2 ([1], [31]) *Let $D>1$ and $p\in\mathcal{P}_{0,\infty}(\mathbb{R}^n)$. Then*

$$\frac{1}{c_0}r^{\frac{n}{p(0)}}\leqslant\left\|\chi_{B(0,Dr)\setminus B(0,r)}\right\|_{p(\cdot)}\leqslant c_0 r^{\frac{n}{p(0)}}\quad\text{for}\quad 0<r\leqslant 1 \tag{2.7}$$

and

$$\frac{1}{c_\infty} r^{\frac{n}{p_\infty}} \leqslant \left\| \chi_{B(0,Dr)\setminus B(0,r)} \right\|_{p(\cdot)} \leqslant c_\infty r^{\frac{n}{p_\infty}} \quad \textit{for} \quad r \geqslant 1, \tag{2.8}$$

respectively, where $c_0 \geqslant 1$ and $c_\infty \geqslant 1$ depend on D, but do not depend on r.

Lemma 2.3 ([31, Lemma 4.7]) *The following relations*

$$\int_{2a<|y|<t} |\Phi(y)|\, dy = \frac{1}{\ln 2} \int_a^t \frac{d\tau}{\tau} \int_{\max(2a,\tau)<|y|<\min(t,2\tau)} |\Phi(y)|\, dy,\ t > 2\alpha > 0, \tag{2.9}$$

and

$$\int_{|y|>2t} |\Phi(y)|\, dy = \frac{1}{\ln 2} \int_t^\infty \frac{d\tau}{\tau} \int_{\max(\tau,2t)<|y|<2\tau} |\Phi(y)|\, dy,\ t > 0, \tag{2.10}$$

hold for every measurable function Φ for which the integrals on the left-hand side exists.

2.3 Riesz potential operator

Recall that the *Riesz potential operator* is given by

$$I^\lambda f(x) = \frac{1}{\gamma_n(\lambda)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\lambda}},\, dy$$

with the normalizing constant $\gamma_n(\lambda) = 2^\lambda \pi^{\frac{n}{2}} \frac{\Gamma(\lambda/2)}{\Gamma((n-\lambda)/2)}$.

Whenever $\lambda p(x)<n$, by p^* we denote the *Sobolev conjugate of p* defined via the usual relation

$$\frac{1}{p^*(x)} := \frac{1}{p(x)} - \frac{\lambda}{n}, \quad x \in \mathbb{R}^n, \tag{2.11}$$

so that $p^* \in \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$, if $p \in \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ and $\lambda p^+ < n$.

The well-known *Sobolev theorem* was extended to variable exponents in [29] for bounded sets in $\mathbb{R}^n$ under the assumption that the maximal operator is bounded in $L^{p(\cdot)}(\Omega)$; for unbounded sets, proved in [3], Sobolev theorem runs as follows.

Theorem 2.4 *Let $p \in \mathscr{P}_\infty^{\log}(\mathbb{R}^n)$ and $\lambda p^+ < n$. Then $\left\| I^\lambda f \right\|_{p^*(\cdot)} \leqslant c \| f \|_{p(\cdot)}$.*

3 Continual variable exponent Herz spaces

Let $\mathbb{R}_{+,\nu} = \{t \in \mathbb{R} : \nu < t < \infty\}$, where $\nu > 0$. By $\mathscr{M}_\infty^{\log}(\mathbb{R}_{+,\nu})$ and $\mathscr{M}_{0,\infty}^{\log}(\mathbb{R}_+)$ we denote the classes of functions g on $\mathbb{R}_{+,\nu}$ and $\mathbb{R}_+$, such that $g(t) = c + g_1(t)$, where c is a constant and $g \in \mathscr{P}_\infty^{\log}(\mathbb{R}_{+,\nu})$ or $g \in \mathscr{P}_{0,\infty}^{\log}(\mathbb{R}_+)$, respectively.

Definition 3.1 We define the *continual variable exponent Herz space*, denoted by $H_{\nu,\delta}^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)$, by the norm

$$\|f\|_{H_{\nu,\delta}^{p(\cdot),q(\cdot),\alpha(\cdot)}(\mathbb{R}^n)} := \|f\|_{L^{p(\cdot)}(B(0,\gamma\nu+\varepsilon))} + \|t^{\alpha(t)} \|f \chi_{\mathsf{R}_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}((\gamma\nu,\infty);dt/t)} < \infty, \tag{3.1}$$

where $0 < \gamma < \delta < \infty$, $\varepsilon > 0$, $q : [\gamma\nu, \infty) \to [1,\ \infty)$, $p : \mathbb{R}^n \to [1,\ \infty)$ and $\alpha : [\gamma\nu,\ \infty) \to \mathbb{R}$ are variable exponents. The cases $\nu = 0$ and $\nu > 0$ correspond to homogeneous and inhomogeneous Herz spaces, respectively. It is known (see Lemma 3.5 in [31]) that this definition is irrelevant to the choice of γ, δ and ε when the exponent q is constant.

The following lemmas were proved in [31], see Lemmas 3.2 and 3.3 there.

Lemma 3.2 *Let $0 < \varepsilon < 2$ and $4 \leqslant R < \infty$. Then*

$$\|f\|_{L^{p(\cdot)}(B(0,R)\setminus B(0,2+\varepsilon))} \leqslant C(\varepsilon, R) \|t^{\alpha(t)} \|f \chi_{t,2t}\|_{L^{p(\cdot)}}\|_{L^q((2,\infty);dt/t)}.$$

Lemma 3.3 *Let* (2.1) *hold. Then the following equivalences of the norms are valid*

$$\|f\|_{H^{p(\cdot),q,\alpha(\cdot)}_{\nu,\delta}} \approx \|f\|_{L^{p(\cdot)}(B(0,\gamma\nu+\varepsilon))} + \|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^q(\mathbb{R}_{+,\nu};dt/t)}, \ \nu > 0, \tag{3.2}$$

and

$$\|f\|_{H^{p(\cdot),q,\alpha(\cdot)}_{0,\delta}} \approx \|t^{\alpha(0)}(1+t)^{\alpha_\infty-\alpha(0)}\|f\chi_{\mathsf{R}_{\gamma t,\delta t}}\|_{L^{p(\cdot)}}\|_{L^{q(\cdot)}(\mathbb{R}_+;dt/t)}, \tag{3.3}$$

if $\alpha \in \mathscr{M}^{\log}_{\infty}(\mathbb{R}_{+,\nu})$ *in the case of* (3.2) *and* $\alpha \in \mathscr{M}^{\log}_{0,\infty}(\mathbb{R}_+)$ *in the case of* (3.3).

4 Main results

In Theorems 4.1 and 4.3 we obtain the conditions of the boundedness of the Riesz potential operator in continual variable Herz space, where all the main parameters $p(\cdot)$, $q(\cdot)$ and $\alpha(\cdot)$ are variable, with different formulation for variable and constant q.

Recall that norms in Herz spaces with constant q are all equivalent to each other for different values of the auxiliary parameters γ and δ, $0 < \gamma < \delta < \infty$, which is not the case when q is variable. So the dependence on those auxiliary parameters come into play when q is variable. The main difference between two cases is in the necessity to move the parameters γ and δ in the starting space when q is variable. By this reason in our main result we slightly change the notation for Herz spaces, taking it in the form

$$H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\nu;(\gamma,\delta)}(\mathbb{R}^n).$$

4.1 Inhomogeneous Herz spaces (the case $\nu = 0$)

Theorem 4.1 *Let* $\nu > 0$, $0 < \gamma < \delta < \infty$, $0 < \lambda < n$, $p \in \mathscr{P}_\infty(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < n/\lambda$, $q \in \mathscr{P}^{\log}_\infty(\mathbb{R}_{\nu+})$ *with* $1 \leqslant q_- \leqslant q_+ < \infty$ *and* $\alpha \in \mathscr{M}_\infty(\mathbb{R}_{\nu+})$. *If*

$$\lambda - \frac{n}{p(\infty)} < \alpha(\infty) < \frac{n}{p'(\infty)}, \tag{4.1}$$

and the Sobolev theorem for the operator I^λ *is valid in variable exponent Lebesgue setting, i.e.* I^λ *is bounded from* $L^{p(\cdot)}(\mathbb{R}^n)$ *to* $L^{p^*(\cdot)}(\mathbb{R}^n)$, *then it it is also bounded from* $H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{\nu;(\gamma',\delta')}(\mathbb{R}^n)$ *to* $H^{p^*(\cdot),q(\cdot),\alpha(\cdot)}_{\nu;(\gamma,\delta)}(\mathbb{R}^n)$ *for any* $0 < \gamma' < \gamma$ *and* $\delta < \delta' < \infty$.

When q is constant, then the same is true also with $\gamma' = \gamma$ *and* $\delta' = \delta$.

P r o o f o f T h e o r e m 4.1. —The case of constant q. We choose $\nu = 2$ and $\gamma = 1, \delta = 2$ in (3.1) for simplicity, so that we will work with the norm

$$\|f\|_{H^{p(\cdot),q,\alpha(\cdot)}_{2,2}} = \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + \mathcal{N}(f)_{p,q,\alpha} \tag{4.2}$$

where

$$\mathcal{N}(f)_{p,q,\alpha} = \|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{L^{p(\cdot)}}\|_{L^q((2,\infty);dt/t)}. \tag{4.3}$$

We first estimate $\left\|I^\lambda f\right\|_{L^{p^*(\cdot)}(B(0,2+\varepsilon))}$, where it suffices to consider $\varepsilon \in (0,1)$. We split f as $f = f\chi_{B(0,8)} + f\chi_{\mathbb{R}^n\setminus B(0,8)} =: f_1 + f_2$ and for $I^\lambda f_1$ we have

$$\left\|I^\lambda f_1\right\|_{L^{p^*(\cdot)}(B(0,2+\varepsilon))} \lesssim \|f\|_{H^{p(\cdot),q,\alpha(\cdot)}_{2,2}}$$

where we have used the $\left(L^{p(\cdot)} \to L^{p^*(\cdot)}\right)$-boundedness of I^λ and Lemma 3.2. For $I^\lambda f_2$ we obtain

$$\begin{aligned}\left|I^\lambda f_2(x)\right| &\lesssim \int_4^\infty \frac{d\tau}{\tau}\int_{\mathsf{R}_{\tau,2\tau}} \frac{|f(y)|}{|x-y|^{n-\lambda}}\,dy\\ &\lesssim \int_4^\infty \tau^{\lambda-n-1}d\tau\int_{\mathsf{R}_{\tau,2\tau}}|f(y)|\,dy\end{aligned}$$

$$\begin{aligned}&\lesssim \int_4^\infty \tau^{-\alpha_\infty+\lambda-n+\frac{n}{p'_\infty}}\left(\tau^{\alpha_\infty}\|f\chi_{\mathsf{R}_{\tau,2\tau}}\|_{L^{p(\cdot)}}\right)\frac{d\tau}{\tau}\\&\lesssim \left\|\tau^{-\alpha_\infty-\frac{n}{p^*_\infty}}\right\|_{L^{q'}((4,\infty);dt/t)}\|f\|_{H_{2,2}^{p(\cdot),q,\alpha(\cdot)}}\\&\lesssim \|f\|_{H_{2,2}^{p(\cdot),q,\alpha(\cdot)}}\end{aligned} \tag{4.4}$$

where the first inequality is due to relation (2.10), the second one comes from the fact that $|x-y|\geqslant |y|-|x|\geqslant \tau/4$ since $x\in B(0,3)$, the third inequality is obtained via Hölder's inequality and estimate (2.8), to obtain the forth inequality we use Hölder's inequality again with the Haar measure and the definition of Sobolev exponent and the last one comes from the fact that the $L^{q'}$-norm of the power function in the forth inequality is finite, which is easily seen by checking the modular.

To estimate now the $\mathcal{N}\big(I^\lambda f\big)_{p^*,q,\alpha}$ term, we split the function $f(x)$ as

$$f(x)=f_0(x)+f_t(x)+g_t(x)+h_t(x),$$

where

$$f_0(x)=f(x)\chi_{B(0,1)}(x),\qquad f_t(x)=f(x)\chi_{B(0,t/2)\setminus B(0,1)}(x),$$

$$g_t(x)=f(x)\chi_{B(0,8t)\setminus B(0,t/2)}(x),\quad h_t(x)=f(x)\chi_{\mathbb{R}^n\setminus B(0,8t)},$$

so that we have the pointwise inequality

$$\left|I^\lambda f(x)\right|\leqslant \left|I^\lambda(f_0)(x)\right|+\left|I^\lambda(f_t)(x)\right|+\left|I^\lambda(g_t)(x)\right|+\left|I^\lambda(h_t)(x)\right|.$$

Estimation of $I^\lambda f_0$. Since $y\in B(0,1)$ and $x\in \mathsf{R}_{t,2t}$ we have that $|x-y|\geqslant |x|-|y|>t/2$, which implies

$$\left|I^\lambda f_0(x)\right|\lesssim t^{\lambda-n}\int_{B(0,1)}|f(y)|\,dy\lesssim t^{\lambda-n}\|f_0\|_{p(\cdot)}\|\chi_{B(0,1)}\|_{p'(\cdot)}\lesssim t^{\lambda-n}\|f_0\|_{p(\cdot)}$$

and we obtain

$$\begin{aligned}\mathcal{N}\big(I^\lambda f_0\big)_{p^*,q,\alpha}&=\|t^{\alpha_\infty}\|\chi_{\mathsf{R}_{t,2t}}I^\lambda f_0\|_{p^*(\cdot)}\|_{L^q((2,\infty);dt/t)}\\&\lesssim \|t^{\alpha_\infty+\lambda-n}\|\chi_{\mathsf{R}_{t,2t}}\|_{p^*(\cdot)}\|_{L^q((2,\infty);dt/t)}\|f_0\|_{p(\cdot)}\\&\lesssim \|t^{\alpha_\infty-\frac{n}{p'_\infty}}\|_{L^q((2,\infty);dt/t)}\|f_0\|_{p(\cdot)}\lesssim \|f_0\|_{p(\cdot)}\end{aligned}$$

where we have used the estimate (2.8), the definition of Sobolev exponent and for the finiteness of the q-norm of the power function in the last line, by relation (4.1).

Estimation of $I^\lambda f_t$. For $x\in\mathsf{R}_{t,2t}$, we have

$$\left|I^\lambda f_t(x)\right|\lesssim \int_{B(0,t/2)\setminus B(0,1)}\frac{|f(y)|}{|x-y|^{n-\lambda}}\,dy.$$

Since $|x-y|\geqslant |x|-|y|\geqslant t/2$, using relation (2.9), Hölder's inequality and (2.8) we obtain

$$\begin{aligned}\left|I^\lambda f_t(x)\right|&\lesssim t^{\lambda-n}\int_{1<|y|<t/2}|f(y)|\,dy\\&\lesssim t^{\lambda-n}\int_1^t\frac{d\tau}{\tau}\int_{\tau/2<|y|<\tau}|f(y)|\,dy\\&\lesssim t^{\lambda-n}\int_1^t\left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)}\left\|\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p'(\cdot)}\frac{d\tau}{\tau}\\&\lesssim t^{\lambda-n}\int_1^t\left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)}\tau^{\frac{n}{p'_\infty}-1}d\tau.\end{aligned} \tag{4.5}$$

We have

$$
\begin{aligned}
t^{\alpha_\infty}\left\|\chi_{\mathsf{R}_{t,2t}} I^\lambda f_t\right\|_{p^*(\cdot)} &\lesssim t^{\alpha_\infty-n+\lambda}\|\chi_{\mathsf{R}_{t,2t}}\|_{p^*(\cdot)}\int_1^t \left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)}\tau^{\frac{n}{p'_\infty}-1}d\tau\\
&\lesssim t^{\alpha_\infty-n+\lambda+\frac{n}{p^*_\infty}}\int_1^t \left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)}\tau^{\frac{n}{p'_\infty}-1}d\tau\\
&\lesssim t^{\alpha_\infty-\frac{n}{p'_\infty}}\int_1^t \left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)}\tau^{\frac{n}{p'_\infty}-1}d\tau\\
&\lesssim \int_1^t \left(\frac{t}{\tau}\right)^{\alpha_\infty-\frac{n}{p'_\infty}}\varphi(\tau)\,\frac{d\tau}{\tau}
\end{aligned}
\tag{4.6}
$$

where $\varphi(\tau)=\tau^{\alpha_\infty}\left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)}$ and the last inequality in the right-hand side of (4.6) is a Hardy type operator. Taking $\mathcal{K}(t)$ as

$$
\mathcal{K}(t)=\begin{cases} t^{\alpha_\infty-\frac{n}{p'_\infty}}, & t>1,\\ 0, & 0<t<1,\end{cases}
\tag{4.7}
$$

we define, as in (2.5), the operator $K\varphi(t)=\int_0^\infty \mathcal{K}(t/\tau)\varphi(\tau)\,\frac{d\tau}{\tau}$. With all the above taken into account we have

$$
\begin{aligned}
\mathcal{N}\big(I^\lambda f_t\big) &\lesssim \|K\varphi\|_{L^q((2,\infty);dt/t)}\\
&\lesssim \|\varphi\|_{L^q((2,\infty);dt/t)}\\
&\lesssim \|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((1,2);dt/t)}+\|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)}\\
&\lesssim \|f\|_{L^{p(\cdot)}(B(0,4)\backslash B(0,1))}+\|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)}\\
&\lesssim \|f\|_{H_2^{p(\cdot),q,\alpha(\cdot)}}
\end{aligned}
$$

where we have used Lemma 2.1, the fact that $\alpha_\infty-n/p'_\infty<0$ and for the last inequality we used the embedding

$$
B(0,4)\backslash B(0,1)\subset B(0,2+\varepsilon)\cup B(0,4)\backslash B(0,2+\varepsilon)
$$

and Lemma 3.2.

ESTIMATION OF $I^\lambda g_t$. By the $\big(L^{p(\cdot)}\to L^{p^*(\cdot)}\big)$-boundedness of I^λ we obtain

$$
\left\|\big(I^\lambda g_t\big)\chi_{\mathsf{R}_{t,2t}}\right\|_{p^*(\cdot)}\lesssim \|g_t\|_{p(\cdot)}\lesssim \sum_{j=-1}^{2}\left\|f\chi_{\mathsf{R}_{2^j t,2^{j+1}t}}\right\|_{p(\cdot)}.
$$

Then

$$
\begin{aligned}
\mathcal{N}\big(I^\lambda g_t\big)_{p^*,q,\alpha} &\lesssim \|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{\frac{t}{2},t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)}+\|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)}\\
&\lesssim \|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((1,2);dt/t)}+\|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)}\\
&\lesssim \|f\|_2^{H^{p(\cdot),q,\alpha(\cdot)}}
\end{aligned}
$$

where the first inequality comes from the fact that

$$
\|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{2^j t,2^{j+1}t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)}\lesssim \|t^{\alpha_\infty}\|f\chi_{\mathsf{R}_{t,2t}}\|_{p(\cdot)}\|_{L^q((2,\infty);dt/t)},\qquad j=1,2,
$$

which follows from a dilation change of variables in t (thanks to the fact that q is constant) and in the last inequality we used Lemma 3.2 and the obvious inequality $\|f\chi_{\mathsf{R}_{t/2,t}}\|_{L^{p(\cdot)}}\leqslant \|f\chi_{\mathsf{R}_{0,4}}\|_{L^{p(\cdot)}}$.

ESTIMATION OF $I^\lambda h_t$. We have

$$
|I^\lambda h_t(x)|\lesssim \int_{|y|>8t}\frac{|f(y)|}{|x-y|^{n-\lambda}}\,dy,
$$

which is estimated as follows

$$\int_{|y|>8t} \frac{|f(y)|}{|x-y|^{n-\lambda}}\,dy \lesssim \int_{4t}^{\infty} \frac{d\tau}{\tau} \int_{\tau}^{2\tau} \frac{|f(y)|}{|x-y|^{n-\lambda}}\,dy$$
$$\lesssim \int_{4t}^{\infty} \tau^{\lambda-n} \|\chi_{\mathsf{R}_{\tau,2\tau}}\|_{p'(\cdot)} \|f\chi_{\mathsf{R}_{\tau,2\tau}}\|_{p(\cdot)} \frac{d\tau}{\tau}$$
$$\lesssim \int_{4t}^{\infty} \tau^{-\frac{n}{p^*_\infty}} \|f\chi_{\mathsf{R}_{\tau,2\tau}}\|_{p(\cdot)} \frac{d\tau}{\tau}$$

where the first inequality comes from relation (2.10), the second by Hölder's inequality and the fact that $|x-y| \geqslant \frac{\tau}{2}\big(\text{since } x \in \mathsf{R}_{t,2t} \text{ there}\big)$ and the last one by relation (2.8). We now have

$$t^{\alpha_\infty} \left\|\chi_{\mathsf{R}_{t,2t}} I^\lambda h_t\right\|_{p^*(\cdot)} \lesssim t^{\alpha_\infty+\frac{n}{p^*_\infty}} \int_{4t}^{\infty} \tau^{-\frac{n}{p^*_\infty}} \|f\chi_{\mathsf{R}_{\tau,2\tau}}\|_{p(\cdot)} \frac{d\tau}{\tau}$$
$$\lesssim \int_t^{\infty} \left(\frac{t}{\tau}\right)^{\alpha_\infty+\frac{n}{p^*_\infty}} \varphi(\tau)\,\frac{d\tau}{\tau}$$

where $\varphi(\tau) = \tau^{\alpha_\infty}\|f\chi_{\mathsf{R}_{\tau,2\tau}}\|_{p(\cdot)}\chi_{(2,\infty)}(\tau)$. We note that we arrived at a Hardy type inequality and since $\alpha_\infty + n/p^*_\infty > 0$, using Lemma 2.1, we obtain

$$\left\|t^{\alpha_\infty}\left\|\chi_{\mathsf{R}_{\tau,2\tau}} I^\lambda h_t\right\|_{p(\cdot)}\right\|_{L^q((2,\infty);dt/t)} \lesssim \|\varphi\|_{L^q((2,\infty);dt/t)} \lesssim \|f\|_{H^{p(\cdot),q,\alpha(\cdot)}_{2,2}}. \qquad \square$$

P r o o f o f T h e o r e m 4.1. —The case of variable q. The principal reason why we cannot act as in the case of constant q, is the fact that, when q is variable, we cannot make a dilation change of variables in t as it was done in the estimation of the term $I^\lambda g_t$. However, the proof follows in the whole the same arguments, so we omit details, but dwell on changes.

We again choose $\nu = 2$ and $\gamma = 1$, $\delta = 2$ for simplicity but stress once again that the norms in the Herz space with variable $q(\cdot)$ are not necessarily equivalent for different values of these parameters. We want to show that

$$\left\|I^\lambda f\right\|_{L^{p^*(\cdot)}(B(0,2+\varepsilon))} + \mathcal{N}^{p^*,q,\alpha}_{1,2}\big(I^\lambda f\big) \lesssim \|f\|_{L^{p(\cdot)}(B(0,2+\varepsilon))} + \mathcal{N}^{p,q,\alpha}_{\gamma',\delta'}(f)$$

with $\gamma' < 1$, $\delta' > 2$ and

$$\mathcal{N}^{p,q,\alpha}_{\gamma,\delta}(g) := \left\|t^{\alpha_\infty}\left\|g\chi_{\mathsf{R}_{\gamma t,\delta t}}\right\|_{L^{p(\cdot)}}\right\|_{L^{q(\cdot)}((2,\infty),dt/t)}.$$

The estimation for $\left\|I^\lambda f\right\|_{L^{p^*(\cdot)}(B(0,2+\varepsilon))}$ is obtained, mutatis mutandis, as in the case of constant q.

To estimate now the $\mathcal{N}\big(I^\lambda f\big)^{p^*,q,\alpha}_{1,2}$ term, we split the function $f(x)$ as

$$f(x) = f_0(x) + f_t(x) + g_t(x) + h_t(x)$$

where

$$f_0(x) = f(x)\chi_{B(0,\frac12)}(x), \qquad f_t(x) = f(x)\chi_{B(0,\gamma' t)\setminus B(0,\frac12)}(x)$$
$$g_t(x) = f(x)\chi_{B(0,\delta' t)\setminus B(0,\gamma' t)}(x), \qquad h_t(x) = f(x)\chi_{\mathbb{R}^n\setminus B(0,\delta' t)}.$$

Then

$$\left|I^\lambda f(x)\right| \leqslant \left|I^\lambda(f_0)(x)\right| + \left|I^\lambda(f_t)(x)\right| + \left|I^\lambda(g_t)(x)\right| + \left|I^\lambda(h_t)(x)\right|.$$

ESTIMATION OF $I^\lambda f_0$. It follows, mutatis mutandis, the one given in the constant q.

ESTIMATION OF $I^\lambda f_t$. By the same estimations of (4.5) and taking into account that $|x-y| > (1-\gamma')t$ we obtain

$$\left|I^\lambda f_t(x)\right| \lesssim t^{\lambda-n} \int_{\frac12}^{t} \left\|f\chi_{\mathsf{R}_{\frac{\tau}{2},\tau}}\right\|_{p(\cdot)} \tau^{\frac{n}{p'_\infty}-1}\,d\tau,$$

from which we get

$$t^{\alpha_\infty} \left\| \chi_{\mathsf{R}_{t,2t}} I^\lambda f_t \right\|_{p^*(\cdot)} \lesssim \int_{\frac{1}{2}}^{t} \left(\frac{t}{\tau}\right)^{\alpha_\infty - \frac{n}{p'_\infty}} \tau^{\alpha_\infty} \left\| f \chi_{\mathsf{R}_{\tau/2,\tau}} \right\|_{p(\cdot)} \frac{d\tau}{\tau}$$
$$\lesssim \int_{\frac{1}{4}}^{t} \left(\frac{t}{\tau}\right)^{\alpha_\infty - \frac{n}{p'_\infty}} \psi(\tau) \frac{d\tau}{\tau} \tag{4.8}$$

where $\psi(\tau) = \tau^{\alpha_\infty} \| f \chi_{\mathsf{R}_{\tau,2\tau}} \|_{p(\cdot)}$. Observing, as in (4.6), that the right-hand side of (4.8) is a Hardy type operator and using Lemma 2.1 together with (4.7) we get

$$\mathcal{N}_{1,2}^{p^*,q,\alpha}\left(I^\lambda f_t\right) \lesssim \|K\psi\|_{L^{q(\cdot)}((2,\infty);dt/t)} \lesssim \|\psi\|_{L^{q(\cdot)}((2,\infty);dt/t)} \lesssim \|f\|_{H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{2;(\gamma',\delta')}(\mathbb{R}^n)},$$

where we followed the same reasoning as in the case of constant q.

ESTIMATION OF $I^\lambda g_t$. By the $\left(L^{p(\cdot)} \to L^{p^*(\cdot)}\right)$-boundedness of I^λ we get

$$\left\| \left(I^\lambda g_t\right) \chi_{\mathsf{R}_{t,2t}} \right\|_{L^{p(\cdot)}} \lesssim \|g_t\|_{L^{p(\cdot)}} = \|f \chi_{\mathsf{R}_{\gamma' t,\delta' t}}\|_{L^{p(\cdot)}}$$

which implies

$$\mathcal{N}_{1,2}^{p^*,q,\alpha}\left(I^\lambda g_t\right) \lesssim \|f\|_{H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

ESTIMATION OF $I^\lambda h_t$. Similarly adapting the estimate of $I^\lambda h_t$ from the proof of the constant q case to the case of variable q, we get

$$\left\| t^{\alpha_\infty} \left\| \chi_{\mathsf{R}_{\tau,2\tau}} I^\lambda h_t \right\|_{p(\cdot)} \right\|_{L^q((2,\infty);dt/t)} \lesssim \|f\|_{H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{2;(\gamma',\delta')}(\mathbb{R}^n)}.$$

Taking all the estimates into account, we obtain the result. □

Corollary 4.2 *The statement of Theorem 4.1 is valid if the assumption that Sobolev theorems holds for variable Lebesgue spaces is replaced by the conditions that $p(x)$ satisfies the log-condition* (2.2) *and* $p_- > 1$.

P r o o f . It suffices to refer to Theorem 2.4. □

4.2 Homogeneous Herz spaces (the case $\nu = 0$)

The main difference in the proof is that now we have to arrange the corresponding splitting of f with respect to not only large values of t, but also near the origin. Correspondingly, the assumptions on all the variable exponents now should include the decay condition not only at infinity, but also at the origin. But in the whole the proofs for this case follow the same lines as in the non-homogeneous case, so we give only the formulation of the result.

Theorem 4.3 *Let* $0 < \gamma < \delta < \infty$, $0 < \lambda < n$, $p \in \mathscr{P}_{0,\infty}(\mathbb{R}^n)$ *with* $1 < p^- \leqslant p^+ < n/\lambda$, $q \in \mathscr{P}^{\log}_{0,\infty}(\mathbb{R}_{\nu+})$ *with* $1 \leqslant q_- \leqslant q_+ < \infty$ *and* $\alpha \in \mathscr{M}_{0,\infty}(\mathbb{R}_+)$. *If*

$$\lambda - \frac{n}{p(0)} < \alpha(0) < \frac{n}{p'(0)} \quad \textit{and} \quad \lambda - \frac{n}{p(\infty)} < \alpha(\infty) < \frac{n}{p'(\infty)}, \tag{4.9}$$

and the Sobolev theorem for the operator I^λ is valid in variable exponent Lebesgue setting, i.e. I^λ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to $L^{p^*(\cdot)}(\mathbb{R}^n)$, then it it is also bounded from $H^{p(\cdot),q(\cdot),\alpha(\cdot)}_{0;(\gamma',\delta')}(\mathbb{R}^n)$ to $H^{p^*(\cdot),q(\cdot),\alpha(\cdot)}_{0;(\gamma,\delta)}(\mathbb{R}^n)$ for any $0 < \gamma' < \gamma$ and $\delta < \delta' < \infty$. When q is constant, then the same is true with $\gamma' = \gamma$ and $\delta' = \delta$.

Corollary 4.4 *The statement of Theorem 4.3 is valid if the assumption that Sobolev theorems holds for variable Lebesgue spaces is replaced by the conditions that $p(x)$ satisfies the log-condition* (2.2) *and* $p_- > 1$.

Remark 4.5 Checking the proofs of Theorem 4.1, we can obtain the same result for sublinear operators T^λ which are $L^{p(\cdot)}(\mathbb{R}^n) \to L^{p^*(\cdot)}(\mathbb{R}^n)$ bounded and satisfy the size condition

$$|T^\lambda f(x)| \lesssim \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\lambda}}\, dy, \qquad x \notin \operatorname{supp} f,$$

where $0 \leqslant \lambda < n$, for integrable and compactly supported functions f.

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