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A NOTE ON THE BEST CONSTANTS IN SOME HARDY INEQUALITIES

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Abstract. The sharp constants in Hardy type inequalities are known only in a few cases. In this paper we discuss some situations when such sharp constants are known, but also some new sharp constants are derived both in one-dimensional and multi-dimensional cases.

1. Introduction

G. H. Hardy stated in 1920 (see [3]) and proved in 1925 (see [4]) his famous inequality: if $f(x)$ is non-negative and measurable on $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \quad p>1. \tag{1.1}$$

Hardy himself presented in 1927 (see [5]) the first generalization of (1.1), namely:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p x^\alpha dx \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)x^\alpha dx, \quad p\geqslant 1, \quad \alpha<p-1. \tag{1.2}$$

The interesting period of more than 10 years of research until Hardy finally discovered his inequalities (1.1) and (1.2) was recently described in [7]. The further development of improvements of (1.1) and (1.2) to what today is called *Hardy type inequality*, is discussed in many books, see e.g. [9], [8] and [6]. First we mention the following modern form of the Hardy original inequality:

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)v(x)dx\right)^{\frac{1}{p}}, \tag{1.3}$$

where $f(x)\geqslant 0$, u and v are weights and $1\leqslant p<\infty$, $0<q<\infty$. In this paper we consider the case $1<p\leqslant q<\infty$. Then it is well known that (1.3) holds if and only if

$$A_{MB} := \sup_x \left(\int_x^\infty u(t)\,dt\right)^{\frac{1}{q}} \left(\int_0^x v^{1-p'}(t)\,dt\right)^{\frac{1}{p}} < \infty, \tag{1.4}$$

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where $p' = \frac{p}{p-1}$, and moreover,

$$A_{MB} \leqslant C \leqslant k(p,q)A_{MB}. \tag{1.5}$$

We remark that many expressions for $k(p,q)$ are known (see [8], pp. 46–47), e.g.

$$k(p,q) = p^{\frac{1}{q}}(p')^{\frac{1}{p}} \quad \text{or} \quad k(p,q) = \left(1+\frac{q}{p'}\right)^{\frac{1}{q}}\left(1+\frac{p'}{q}\right)^{\frac{1}{p'}}.$$

We also remark that a nice proof of the characterization (1.4) was given in 1972 by B. Muckenhoupt [11] for the case $p = q$ and in 1978 by J. S. Bradley [2] for the general case $1 < p \leqslant q < \infty$. For the case $p = q$ we also refer to earlier papers by G. Talenti [13] and G. A. Tomaselli [14].

The constants in (1.1) and (1.2) are sharp, but for the best constant C in (1.3) we can only give an estimate of the type (1.5). In this paper we will consider the simpler case with power weights where indeed the sharp constant can be found (see Theorem 3.1). First we note that by applying (1.4) to the case of power weights, one easily obtains the following (see Example 0.3(i) in [9]):

EXAMPLE 1.1. The inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)x^\beta dx\right)^{\frac{1}{p}} \tag{1.6}$$

holds for $1 < p \leqslant q < \infty$, if and only if

$$\beta < p-1 \quad \text{and} \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1. \tag{1.7}$$

In (1.3) we can consider the situation with $\int_0^x$ replaced by $\int_x^\infty$ and this inequality can be characterized by a condition similar to that in (1.4). For the special case of power weights we have (see Example 0.3 (ii) in [9]):

EXAMPLE 1.2. The inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^\alpha dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty f^p(x)x^\beta dx\right)^{\frac{1}{p}} \tag{1.8}$$

holds for $1 < p \leqslant q < \infty$, if and only if

$$\beta > p-1 \quad \text{and} \quad \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1.$$

In Section 2 we present some preliminaries of independent interest:

(a) G. A. Bliss [1] pointed out the sharp constant in (1.6) for the case $\beta = 0$.

(b) V. M. Manakov [10] proved an important result connected to the best constant for the general case (1.3).

(c) We prove that the inequalities (1.6) and (1.8) are in a sense equivalent.

By using mainly these results and ideas, we derive the sharp constants in (1.6) and (1.8) for all the considered cases, see Section 3.

In Section 4 we prove some corresponding new multidimensional Hardy type inequalities.

2. Preliminaries

2.1. The Bliss result

For the case $\beta = 0$, G. A. Bliss [1] proved the following result:

THEOREM 2.1. *Let* $1 < p < q < \infty$. *The inequality* (1.6) *with* $\beta = 0$ *and* $\alpha = -\frac{q}{p'} - 1$ *holds with the sharp constant*

$$C_{pq} = \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{qp}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}} \tag{2.1}$$

and with this constant (1.6) *turns into equality if and only if*

$$f(x) = \frac{C}{\left(ax^{\frac{q}{p}-1}+1\right)^{\frac{q-p}{p}}}. \tag{2.2}$$

Next we point out that there is a continuity in the sharp constant (2.1) as $q \to p$, a fact not pointed out in the original paper of Bliss.

REMARK 2.2. There holds

$$C_{pq} \to \frac{p}{p-1}$$

as $q \to p$.

In fact, by using the Stirling formula

$$\Gamma(x) \approx \sqrt{\frac{2\pi}{x}}\left(\frac{x}{e}\right)^x \quad \text{as} \quad x \to \infty$$

and using the notation $\mu = \frac{qp}{q-p}$ we find that

$$C_{pq} \approx \left(\frac{p}{q(p-1)}\right)^{\frac{1}{p}} q^{\frac{1}{\mu}} \left(\frac{1}{\mu}\right)^{\frac{1}{\mu}} \left(\frac{\sqrt{\frac{2\pi}{\mu}} \left(\frac{\mu}{e}\right)^{\mu}}{\sqrt{\frac{2\pi}{\mu/q}} \left(\frac{\mu}{qe}\right)^{\frac{\mu}{q}} \sqrt{\frac{2\pi}{\mu/q'}} \left(\frac{\mu}{q'e}\right)^{\frac{1}{\mu}}} \right)^{\frac{1}{\mu}}$$

$$\approx \left(\frac{1}{p-1}\right)^{\frac{1}{p}} p^{\frac{1}{p}} (p')^{\frac{1}{p'}} = \frac{p}{p-1} \text{ as } \mu \to \infty \text{ (q} \to \text{p)}$$

and it remains to note that $\frac{p}{p-1}$ is the sharp constant in the Hardy original inequality (1.1).

2.2. The Manakov result

By using in particular the Bliss result, V. M. Manakov [10] proved the following:

THEOREM 2.3. *Let $1 < p < q < \infty$ and v be a weight function such that $\int_0^\infty v^{1-p'}(x)dx = \infty$. Then there exists a weight function u such that $0 < A_{MB} < \infty$ and in this case the sharp constant $C = C_{pq}$ in the weighted Hardy inequality* (1.3) *is determined by*

$$C_{pq} = \left(\frac{\Gamma\left(\frac{qp}{q-p}\right)}{\Gamma\left(\frac{q}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)} \right)^{\frac{1}{p}-\frac{1}{q}} A_{MB}.$$

REMARK 2.4. Manakov pointed out a constant for the power weight case corresponding to (in our notation) the special case $\beta = -\frac{2p-q}{q-p}(p-1)$ and correspondingly $\alpha = -\frac{p(q-1)}{q-p}$ (see formula (16) in [10]). Compare with our Theorem 3.1

2.3. Some equivalence results

In the case $p = q$, so that $\alpha = \beta - p$, we have the following precise information:

THEOREM 2.5. *Let f be a measurable non-negative function on $(0,\infty)$ and let $p \geqslant 1$.*

a) The inequality (1.6) *with $q = p$ holds with $\alpha = \beta - p$ when $\beta < p-1$, with the sharp constant $C = \frac{p}{p-1-\beta}$.*

b) The inequality (1.8) *with $q = p$ and with β replaced by β_0 holds when $\beta_0 > p-1$, and $\alpha = \beta_0 - p$ with the sharp constant $C = \frac{p}{\beta_0+1-p}$.*

c) The inequalities (1.6) *and* (1.8) *(with β replaced by β_0) for $q = p$ are equivalent with the relation $\beta = 2p - \beta_0 - 2$ between the parameters.*

REMARK 2.6. A simple proof of a more general statement may be found in the recent paper [12].

Our next step is to state an equivalence like in Theorem 2.5 for the case $1 < p < q < \infty$.

THEOREM 2.7. *Let $1 < p \leqslant q < \infty$. The following statements (a) and (b) hold and are equivalent:*

(a) The inequality

$$\left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{1/q} \leqslant C\left(\int_0^x f^p(x)x^\beta dx\right)^{1/p} \tag{2.3}$$

holds for all measurable functions $f(t)$ on $(0,\infty)$ if and only if

$$\beta < p-1 \text{ and } \frac{\alpha+1}{q} = \frac{\beta+1}{p} - 1. \tag{2.4}$$

(b) The inequality

$$\left(\int_0^\infty \left(\int_x^\infty f(t)dt\right)^q x^{\alpha_0} dx\right)^{1/q} \leqslant C\left(\int_0^\infty f^p(x)x^{\beta_0} dx\right)^{1/p} \tag{2.5}$$

holds for all measurable functions $f(t)$ on $(0,\infty)$ if and only if

$$\beta_0 > p-1, \frac{\alpha_0+1}{q} = \frac{\beta_0+1}{p} - 1. \tag{2.6}$$

Moreover, it yields that

(c) the formal relation between the parameters β and β_0 is $\beta_0 = -\beta - 2 + 2p$ and in this case the best constants in (2.3) *and* (2.5) *are the same.*

Proof. The fact that (a) and (b) hold is pointed out in Examples 1.1 and 1.2, respectively. We apply the inequality (2.3) with $f(t)$ replaced by $f(1/t)$ and make some obvious substitutions to find that

$$\left(\int_0^\infty \left(\int_u^\infty g(s)ds\right)^q u^{-\alpha-2} du\right)^{1/q} \leqslant C\left(\int_0^\infty g^p(u)u^{2p-\beta-2} du\right)^{1/p},$$

where $g(s) = f(s)/s^2$. We now put $\alpha_0 = -\alpha - 2$ and $\beta_0 = -\beta - 2 + 2p$ and note that, by (2.4), $\beta_0 = -\beta - 2 + 2p > 1 - p - 2 + 2p = p - 1$ and $\alpha_0 = -\alpha - 2 = -\beta\frac{q}{p} + \frac{q}{p'} + 1 - 2 = (\beta_0 + 2 - 2p)\frac{q}{p} + \frac{q}{p'} - 1 = \beta_0\frac{q}{p} - 2\frac{q}{p'} + \frac{q}{p'} - 1 = \beta_0\frac{q}{p} - \frac{q}{p'} - 1$ so that (2.6) also holds.

This means that $(a) \Rightarrow (b)$ but since all calculations are just described by equalities they can be reversed so we have in fact proved that $(a) \Leftrightarrow (b)$.

The statement in (c) holds since all calculations in our proof only consists of equalities. The proof is complete. □

3. The sharp constants in (2.3) and (2.5)

First we use similar ideas as in the proof of the Manakov result and the Bliss result to derive a new constant $C = C^*_{pq}$, which is sharp in (2.3) for each $p \in (1,q)$, and also has the continuity property that C^*_{pq} tends to $\frac{p}{p-1-\beta}$ as $q \to p$, which is the sharp constant for the case $q = p$ by Theorem 2.5.

THEOREM 3.1. *Let $1 < p < q < \infty$ and the parameters α and β satisfy* (2.4)*. Then the sharp constant in* (2.3) *is $C = C^*_{pq}$, where*

$$C^*_{pq} = \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}. \tag{3.1}$$

Equality in (2.3) *occurs exactly when* $f(x) = \dfrac{cx^{-\frac{\beta}{p-1}}}{\left(dx^{\frac{p-1-\beta}{p-1}\cdot\left(\frac{q}{p}-1\right)}+1\right)^{\frac{q}{q-p}}}$. *Moreover,*

$$C^*_{pq} \to \frac{p}{p-1-\beta} \quad \text{as} \quad q \to p. \tag{3.2}$$

Proof. First we make a change of variables in (2.3) by putting

$$s = s(x) = \int_0^x t^{-\frac{\beta}{p-1}} dt = \frac{p-1}{p-1-\beta} x^{\frac{p-1-\beta}{p-1}} \tag{3.3}$$

and define

$$g(s) = g(s(x)) = f(x) x^{\frac{\beta}{p-1}}. \tag{3.4}$$

Then

$$\int_0^\infty f^p(x) x^\beta dx = \int_0^\infty f^p(x) x^{\frac{\beta p}{p-1}} x^{-\frac{\beta}{p-1}} dx = \int_0^\infty g^p(s) ds \tag{3.5}$$

and

$$I_0 := \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^\alpha dx\right)^{1/q} = \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^{\frac{\beta q}{p}-\frac{q}{p'}-1} dx\right)^{1/q}$$
$$= \left(\int_0^\infty \left(\int_0^x f(t)dt\right)^q x^{\frac{\beta q}{p}-\frac{q}{p'}-1+\frac{\beta}{p-1}} x^{-\frac{\beta}{p-1}} dx\right)^{1/q}.$$

Hence, since $f(t)dt = f(t)t^{\frac{\beta}{p-1}} t^{\frac{-\beta}{p-1}} dt = g(r)dr$ and $x = \left(\frac{p-1-\beta}{p-1}s\right)^{\frac{p-1}{p-1-\beta}}$ we have that

$$I_0 = \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\int_0^\infty \left(\int_0^s g(r)dr\right)^q s^{\frac{p-1}{p-1-\beta}\frac{\beta-p+1}{p-1}\left(\frac{q}{p'}+1\right)} ds\right)^{1/q} \tag{3.6}$$
$$= \left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\int_0^\infty \left(\int_0^s g(r)dr\right)^q s^{\frac{-q}{p'}-1} ds\right)^{1/q}.$$

Combining (3.5), (3.6) and using Bliss result (Theorem 2.1) we find that the sharp constant in (2.3) is $C^*_{p,q}$ as defined in (3.1). From the Bliss result (see [2]) it also follows that equality in (2.3) yields exactly when

$$g(s) = \frac{c}{(ds^{\frac{q}{p}-1}+1)^{\frac{q}{q-p}}}$$

i.e. when (see (3.3) and (3.4))

$$f(x) = \frac{cx^{-\frac{\beta}{(p-1)}}}{\left(dx^{\frac{p-1-\beta}{p-1}(\frac{q}{p}-1)}+1\right)^{\frac{q}{q-p}}}.$$

It only remains to prove the relation (3.2). First we note that, according to Remark 2.2,

$$\frac{C^*_{pq}}{\left(\frac{p-1}{p-1-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}}} \to \frac{p}{p-1} \quad \text{as} \quad q \to p$$

so that

$$C^*_{pq} \to \frac{p-1}{p-1-\beta}\cdot\frac{p}{p-1} = \frac{p}{p-1-\beta} \quad \text{as} \quad q \to p.$$

The proof is complete. □

By using this result and Theorem 2.7 we obtain the following sharp constant in (2.5):

THEOREM 3.2. *The sharp constant in* (2.5) *for the case* $1 < p < q < \infty$ *is* $C^\sharp_{p,q}$, *where* $C^\sharp_{p,q}$ *coincides with the constant* $C^*_{p,q}$ *with* β *replaced by* $-\beta_0 - 2 + 2p$.

Equality in (2.5) *occurs if and only if* $f(x)$ *is in the form*

$$f(x) = \frac{cx^{\beta_0/p-1}}{(dx^{(\frac{\beta_0+1-p}{p-1})(\frac{q}{p}-1)}+1)^{\frac{q}{q-p}}} \quad \text{a.e..}$$

Moreover, we have the continuity between sharp constants when $q \to p$, *i.e.*

$$C^\sharp_{p,q} \to \frac{p}{\beta_0+1-p} \quad \text{as} \quad q \to p.$$

4. Some multidimensional Hardy-type inequalities with sharp constants

Let $n \in \mathbb{Z}_+$ and $|S^{n-1}|$ denote the measure of the unit sphere in $\mathbb{R}^n$. First we state the following Hardy-type inequality:

THEOREM 4.1. *Let* $1 < p < q < \infty$. *The inequality*

$$\left(\int_{\mathbb{R}^n}\left(\int_{|y|<|x|} f(y)dy\right)^q |x|^\alpha dx\right)^{\frac{1}{q}} \leqslant C^*_{pq}(n)\left(\int_{\mathbb{R}^n} |f(x)^p||x|^\beta dx\right)^{\frac{1}{p}}, \qquad (4.1)$$

holds if and only if

$$\beta < n(p-1) \quad \text{and} \quad \frac{\alpha+n}{p} = \frac{\beta+n}{q} - n,$$

and in this case it holds with the sharp constant

$$C^*_{pq}(n) = |\mathbb{S}^{n-1}|^{1+\frac{1}{q}-\frac{1}{p}} \left(\frac{p-1}{n(p-1)-\beta}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}$$

when $q > p$, *and*

$$C^*_{pp}(n) = \lim_{q\to p} C^*_{pq}(n) = \frac{p|\mathbb{S}^{n-1}|}{n(p-1)-\beta} \tag{4.2}$$

in the case $q = p$.

Moreover, a stronger inequality holds:

$$\left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{|y|<|x|} f(y)dy\right)^q |x|^\alpha dx\right)^{\frac{1}{q}} \tag{4.3}$$

$$\leqslant C^*_{pq}(n)|\mathbb{S}^{n-1}|^{-\frac{1}{p'}} \left(\int\limits_0^\infty \rho^{n-1+\beta}\left(\int\limits_{\mathbb{S}^{n-1}} f(\rho\sigma)d\sigma\right)^p d\rho\right)^{\frac{1}{p}}.$$

Proof. We find it convenient to rewrite the sharp one'dimensional Hardy inequality (2.3) in Theorem 2.7 with both the weights on the left-hand side, i.e.

$$\left(\int\limits_0^\infty \left|x^{\delta+\gamma-1}\int\limits_0^x \frac{g(t)}{t^\gamma}dt\right|^q dx\right)^{\frac{1}{q}} \leqslant C^*_{p,q,\gamma}\left(\int\limits_0^\infty g^p(x)dx\right)^{\frac{1}{p}}, \quad \gamma < \frac{1}{p'}, \tag{4.4}$$

where $1 < p < q < \infty$ and $\delta = \frac{1}{p} - \frac{1}{q}$ – after the change of notation: $\alpha = (\delta+\gamma-1)q$, $\beta = \gamma p$ and $f(t)t^\gamma = g(t)$ in (2.3); $C^*_{p,q,\gamma}$ is the sharp constant from (3.1) with β in (3.1) replaced by γp. Correspondingly, in the multi-dimensional case, we will deal in the proof with the weighted Hardy operator in the form

$$|x|^{\lambda+\mu-n}\int\limits_{|y|<|x|} \frac{f(y)}{|y|^\mu}dy, \quad \lambda \geqslant 0,$$

where, as expected, λ will be related to p and q by the relation $\frac{\lambda}{n} = \frac{1}{p} - \frac{1}{q}$.

By making a polar coordinate transformation in $\mathbb{R}^n$ we find that

$$\left(\int\limits_{\mathbb{R}^n}\left||x|^{\lambda+\mu-n}\int\limits_{|y|<|x|}\frac{f(y)}{|y|^{\mu}}dy\right|^q dx\right)^{\frac{1}{q}}$$

$$=|\mathbb{S}^{n-1}|^{\frac{1}{q}}\left(\int\limits_0^\infty\left|r^{\frac{n-1}{q}+\lambda+\mu-n}dr\int\limits_0^r\rho^{\frac{n-1}{p'}-\mu}\cdot\rho^{\frac{n-1}{p}}F(\rho)d\rho\right|^q\right)^{\frac{1}{q}},$$

where

$$F(\rho)=\int_{\mathbb{S}^{n-1}}f(\rho\sigma)d\sigma.$$

With the notation

$$\delta=\frac{\lambda}{n}\quad\text{and}\quad\gamma=\mu-\frac{n-1}{p'}$$

we rewrite the above identity as

$$\left(\int\limits_{\mathbb{R}^n}\left||x|^{\lambda+\mu-n}\int\limits_{|y|<|x|}\frac{f(y)}{|y|^{\mu}}dy\right|^q dx\right)^{\frac{1}{q}}=|\mathbb{S}^{n-1}|^{\frac{1}{q}}\left(\int\limits_0^\infty\left|r^{\delta+\gamma-1}dr\int\limits_0^r\rho^{-\gamma}\cdot\rho^{\frac{n-1}{p}}F(\rho)d\rho\right|^q\right)^{\frac{1}{q}}.$$

Note that

$$\mu<\frac{n}{p'}\iff\gamma<\frac{1}{p'},$$

so that the one-dimensional Hardy inequality (4.4) is applicable on the right-hand side with respect to the function $\rho^{\frac{n-1}{p}}F(\rho)$ and we obtain:

$$\left(\int\limits_{\mathbb{R}^n}\left||x|^{\lambda+\mu-n}\int\limits_{|y|<|x|}\frac{f(y)}{|y|^{\mu}}dy\right|^q dx\right)^{\frac{1}{q}}\leqslant C^*_{p,q,\gamma}|\mathbb{S}^{n-1}|^{\frac{1}{q}}\left(\int\limits_0^\infty\rho^{n-1}F^p(\rho)d\rho\right)^{\frac{1}{p}}. \quad (4.5)$$

Moreover, by the Hölder inequality, we have

$$F^p(\rho)=\left(\int\limits_{\mathbb{S}^{n-1}}|f(\rho\sigma)|d\sigma\right)^p\leqslant|\mathbb{S}^{n-1}|^{p-1}\int\limits_{\mathbb{S}^{n-1}}|f(\rho\sigma)|^p d\sigma$$

so that

$$\left(\int\limits_{\mathbb{R}^n}\left||x|^{\lambda+\mu-n}\int\limits_{|y|<|x|}\frac{f(y)}{|y|^{\mu}}dy\right|^q dx\right)^{\frac{1}{q}}\leqslant C^*_{p,q,\gamma}|\mathbb{S}^{n-1}|^{1+\frac{1}{q}-\frac{1}{p}}\left(\int\limits_0^\infty\rho^{n-1}\int\limits_{\mathbb{S}^{n-1}}|f(\rho\sigma)|^p d\sigma d\rho\right)^{\frac{1}{p}}$$

$$=|\mathbb{S}^{n-1}|^{1+\frac{1}{q}-\frac{1}{p}}C^*_{p,q,\gamma}\left(\int\limits_{\mathbb{R}^n}|f(y)|^p dy\right)^{\frac{1}{p}}$$

with $\gamma = \frac{\beta}{p} - \frac{n-1}{p'}$. This yields (4.1) after replacing $\frac{f(y)}{|y|^{\mu}}$ by $f(y)$ and changing the notation:

$$(\lambda + \mu - n)q = \alpha \quad \text{and} \quad \mu p = \beta.$$

The derivation of the constant $C^{*}_{p,p}$ is similar by just using Theorem 2.5 a) instead of Theorem 3.1. The continuity relation in (4.2) can be derived by using some straightforward calculations as in Remark 2.2.

As regards a stronger version of (4.3), it was already obtained by passing in (4.5) in other notation. To see that (4.3) is indeed stronger than (4.1), it suffices to choose for instance $f(\rho\sigma) = A(\rho)B(\sigma)$ where $B(\sigma)$ is just in $L^1(\mathbb{S}^{n-1})$, not necessarily in $L^p(\mathbb{S}^{n-1})$.

Note that the necessity of the relation between α and β is easily obtained from homogeneity arguments. The proof is complete. $\square$

The following result for the dual operator is also valid:

THEOREM 4.2. *Let $1 < p \leqslant q < \infty$. The inequality*

$$\left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{|y|>|x|} f(y)dy\right)^{q} |x|^{\alpha} dx\right)^{\frac{1}{q}} \leqslant C^{\sharp}_{p,q}(n) \left(\int\limits_{\mathbb{R}^n} |f(x)^{p}||x|^{\beta} dx\right)^{\frac{1}{p}}, \tag{4.6}$$

holds if and only if

$$\beta > n(p-1) \quad \text{and} \quad \frac{\alpha+n}{p} = \frac{\beta+n}{q} - n,$$

and in this case it holds with the sharp constant

$$C^{\sharp}_{p,q}(n) = |\mathbb{S}^{n-1}|^{1+\frac{1}{q}-\frac{1}{p}} \left(\frac{p-1}{\beta - n(p-1)}\right)^{\frac{1}{p'}+\frac{1}{q}} \left(\frac{p'}{q}\right)^{\frac{1}{p}} \left(\frac{\frac{q-p}{p}\Gamma\left(\frac{pq}{q-p}\right)}{\Gamma\left(\frac{p}{q-p}\right)\Gamma\left(\frac{p(q-1)}{q-p}\right)}\right)^{\frac{1}{p}-\frac{1}{q}}$$

when $q > p$ and

$$C^{\sharp}_{p,p}(n) = \lim_{q \to p+0} C^{\sharp}_{p,q}(n) = \frac{p|\mathbb{S}^{n-1}|}{\beta - n(p-1)}$$

in the case $q = p$.

Moreover, a stronger inequality holds:

$$\left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{|y|>|x|} f(y)dy\right)^{q} |x|^{\alpha} dx\right)^{\frac{1}{q}} \tag{4.7}$$

$$\leqslant C^{\sharp}_{p,q}(n)|\mathbb{S}^{n-1}|^{-\frac{1}{p'}} \left(\int\limits_0^{\infty} \rho^{n-1+\beta} \left(\int\limits_{\mathbb{S}^{n-1}} f(\rho\sigma)d\sigma\right)^{p} d\rho\right)^{\frac{1}{p}}.$$

Proof. The proof is obtained by using arguments similar to those in the proof of Theorem 4.1, via the applications of the one'dimensional Theorems 2.5 b) and 3.2 in the radial variable, we leave the details for the reader.

Note also that it may be reduced to Theorem 4.1 either by the inversion change of variables

$$y = \frac{z}{|z|^2}, \quad z = \frac{y}{|y|^2}, \quad dy = \frac{dz}{|z|^{2n}}$$

in $\mathbb{R}^n$ with the subsequent change of notation for the function and parameters, or by treating the Hardy inequality as the boundedness of the Hardy operator and passing to the boundedness of the conjugate operator in the dual space. □

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