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Fractional powers of the operator $-|x|^2\Delta$ in L_p-spaces

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Abstract

We investigate fractional powers of the operator $-|x|^2\Delta$ within the framework of the spaces $L_p(\mathbf{R}^n)$. Negative powers are realized as Riemann-Liouville fractional integrals of a strongly continuous semi-group, while positive powers, inverse to negative, are constructed as Marchaud fractional derivatives of that semi-group.

1 Introduction

We consider fractional powers of the operator

$$-|x|^2\Delta, \tag{1.1}$$

where Δ is the Laplace operator in $\mathbf{R}^n$. There are a number of papers devoted to investigation of fractional powers of second order translation invariant differential operators, in particular, the classical operators of mathematical physics: the wave operator, the Klein-Gordon-Fock and Schrödinger operators, the telegraph operator, and others (for references and detailed analysis of the results we refer to the books [16]-[17] and the survey papers [8],[9],[15]. We observe that positive fractional powers $(-\Delta)^{\alpha/2}f$, $\alpha > 0$, of the operator $-\Delta$ are known to be realized as hypersingular integrals (HSI), see the papers [13]-[14] and the books [16]-[17]. The case of multidimensional differential operators, invariant with respect to dilations (and rotations) is almost untouched. We can only mention the papers [4] and [1], devoted to the investigation of fractional powers

$$\left(-\sum_{k=1}^{n}\left(x_k\frac{\partial}{\partial x_k}\right)^2\right)^{\alpha/2} \quad \text{and} \quad \left(\pm\sum_{k=1}^{n}x_k^2\frac{\partial^2}{\partial x_k^2}\right)^{\alpha/2}.$$

The goal of this paper is to fill in this gap with respect to the operator (1.1). Its fractional powers are defined as follows. We base ourselves on the expansion

$$(-|x|^2\Delta\varphi)(x) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{\mu=1}^{d_n(m)}\mathcal{Y}_{m\mu}(x')\int_{\mathrm{Re}\, s=\varkappa}|x|^{-s}\times$$
$$\times(m+n-2-s)(m+s)(\mathfrak{M}\varphi_{m\mu})(s)\,ds, \qquad x\in\mathbf{R}^n, \tag{1.2}$$

where $\{\mathcal{Y}_{m\mu}(x')\}$ is the orthonormal basis of spherical harmonics,

$$(\mathfrak{M}\varphi_{m\mu})(s) = \int_0^\infty \tau^{s-1}\varphi_{m\mu}(\tau)\, d\tau$$

is the Mellin transform of the Fourier-Laplace coefficients

$$\varphi_{m\mu}(r) = \int_{S^{n-1}} \varphi(r\sigma)\mathcal{Y}_{m\mu}(\sigma)\, d\sigma\ .$$

In view of (1.2) it is natural to define the negative powers $(-|x|^2\Delta)^{-\alpha/2}\varphi$, $\mathtt{Re}\,\alpha > 0$, as follows:

$$(-|x|^2\Delta)^{-\alpha/2}\varphi(x) \equiv (\mathbf{I}^\alpha_\varkappa\varphi)(x) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')\int_{\mathtt{Re}\,s=\varkappa} |x|^{-s} \times$$
$$\times(m+n-2-s)^{-\alpha/2}(m+s)^{-\alpha/2}(\mathfrak{M}\varphi_{m\mu})(s)\, ds, \qquad \varphi\in C^\infty_{0,0}. \tag{1.3}$$

As is shown in this paper, the operator $\mathbf{I}^\alpha_\varkappa$ is extended as a bounded operator to the whole space L_p, $1 < p < \infty$. We obtain the integral representation for the potential $\mathbf{I}^\alpha_\varkappa f$, $\mathtt{Re}\,\alpha > 0$, $\varphi \in L_p$, via the Riemann-Liouville fractional integral of a certain strongly continuous semi-group T_t and prove the inversion formula

$$\mathbf{D}^\alpha\mathbf{I}^\alpha_\varkappa\varphi = \varphi,$$

where $\mathbf{D}^\alpha f$ is the Marchaud-type fractional derivative of the semi-group T_t (a realization of the Balakrishnan formula). In fact, we obtain an explicit expression for the positive powers $(-|x|^2\Delta)^{\alpha/2}f$, $\mathtt{Re}\,\alpha > 0$.

The paper is organized as follows. Section 2 contains some auxiliary information, in particular, for radial-spherical expansions and multidimensional integral operators with homogeneous kernels. In Section 3 we define the negative powers $\mathbf{I}^\alpha\varphi = (-|x|^2\Delta)^{-\alpha/2}\varphi$, $\mathtt{Re}\,\alpha > 0$, by (1.3) and prove that the operator $\mathbf{I}^\alpha_\varkappa$ is bounded in $L_p(R^n)$, $1 < p < \infty$. We also obtain the following representation for the potentials $\mathbf{I}^\alpha_\varkappa\varphi$ in terms of radial-spherical convolutions:

$$(\mathbf{I}^\alpha_\varkappa f)(x) = \int_{\mathbf{R}^n} \varphi(y)k_\alpha\left(\frac{|x|}{|y|}, x'\cdot y'\right)\frac{dy}{|y|^n}, \tag{1.4}$$

which is of special interest itself. In Section 4 we study the semi-group T_t generated by the infinitesimal operator $|x|^2\Delta$ and obtain the representation

$$(\mathbf{I}^\alpha_\varkappa f)(x) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}(T_t\varphi)(x)\, dt, \quad \varphi\in L_p \tag{1.5}$$

for the potential $\mathbf{I}^\alpha_\varkappa f$. The equality (1.5) enables us to apply the HSI's method for inverting the potential $f = \mathbf{I}^\alpha\varphi$, $\varphi \in L_p$, provided that $n/(n-2) < p < \infty$. Within the framework of this approach, in Section 5 the inversion is constructed as follows:

$$\mathbf{D}^\alpha f = \overset{(L_p)}{\lim_{\varepsilon\to 0}} \mathbf{D}^\alpha_\varepsilon f, \tag{1.6}$$

where

$$(\mathbf{D}^\alpha_\varepsilon f)(x) = \frac{1}{\varkappa(\alpha/2,l)}\int_\varepsilon^\infty (E-T_\tau)^l f(x)\frac{d\tau}{\tau^{\alpha/2+1}}.$$

Some of the results presented in this paper were announced in [2].

2 Preliminaries

1°. Notation:

$x' = \frac{x}{|x|}$, $x \in R^n$;
S_{n-1} is the unit sphere in R^n;
$|S_{n-1}|$ is its Lebesgue measure;
$d_n(m) = (n+2m-2)\frac{(n+m-3)!}{m!(n-2)!}$,$n > 2$, is the dimension of the space of spherical harmonic of order m;
$(\mathcal{F}\varphi)(\xi) = \int_{\mathbf{R}^n} \varphi(x) \exp(ix\cdot\xi)\, dx$ is the Fourier transform of φ;
$L_p = L_p(\mathbf{R}^n)$; $\|\varphi\|_p = \|\varphi\|_{L_p}$;
$L_p(A, w) = \{f : \|f\|^p_{L_p(A,w)} = \int_A |f(x)|^p w(x)\, dx < \infty\}, w \geq 0$;
$C^\infty_{0,0}(\mathbf{R}_+)$ is the class of infinitely differentiable functions on $\mathbf{R}_+$ with a compact support beyond the origin;
$\mathbf{Z}^0_+ = \{0, 1, 2, \dots\}$;
$\lambda_- = \begin{cases} 0, & \lambda > 0, \\ |\lambda|, & \lambda < 0; \end{cases}$

By z^α, $\alpha \in \mathbf{C}$, we denote the main branch ($k = 0$) of the multi-valued function $z^\alpha = \exp(\alpha[\ln|z| + i\arg z + 2k\pi i])$ analytic in the complex plane cutted along the negative real semiaxis .

The end of the proof is denoted by ■ .

2°. Some properties of the Mellin transform (see [7, § 2]). Let $L^c(0, \infty)$ be the class of complex valued piece-wise continuous functions $f(x)$ on $(0, +\infty)$ with a finite number of jumps and such that $\int_0^\infty |f(x)|\, dx < \infty$. The following statements are valid.

Lemma 2.1 *Let $\varphi(x)x^{\varkappa-1} \in L^c(0, \infty)$ and let $\varphi(x)$ be piece-wise differentiable in some neighborhood $(x_0 - \varepsilon, x_0 + \varepsilon)$ of a point $x_0 > 0$ and continuous at x_0. Then*

$$\varphi(x_0) = \frac{1}{2\pi i} \lim_{\lambda\to\infty} \int_{\varkappa - i\lambda}^{\varkappa + i\lambda} (\mathfrak{M}\varphi)(s) x_0^{-s} ds.$$

Lemma 2.2 *If $f(x)x^{\varkappa-1}$ and $g(x)x^{\varkappa-1} \in L^c(0, \infty)$, then the Mellin convolution $h(x) = \int_0^\infty f(y)g\left(\frac{x}{y}\right)\frac{dy}{y}$ exists and $h(x)x^{\varkappa-1} \in L^c(0, \infty)$, and $(\mathfrak{M}h)(s) = (\mathfrak{M}f)(s)(\mathfrak{M}g)(s)$, where $\mathrm{Re}\, s = \varkappa$.*

3°. Radial-spherical expansions and multipliers (see [12]). Let $\varphi \in C^\infty_{0,0}$. The expansion

$$\varphi(x) = \frac{1}{2\pi i} \sum_{m=0}^\infty \sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x') \int_{\mathrm{Re}\ s=\varkappa} |x|^{-s} (\mathfrak{M}\varphi_{m\mu})(s)\, ds,$$

where the series in the right-hand side converges absolutely and uniformly in any layer $0 < \alpha \leq |x| \leq \beta < \infty$ for all $\varkappa \in \mathbf{R}$, is known as the radial-spherical expansion.

Consider the operator

$$(A_\varkappa\varphi)(x) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x') \times$$
$$\times \int_{\mathrm{Re}\, s=\varkappa} |x|^{-s}\mu_m(s)(\mathfrak{M}\varphi_{m\mu})(s)\, ds, \qquad \varphi \in C^\infty_{0,0},$$

generated by the sequence of measurable functions $\mu_m(s)$, defined on the straight line $\mathrm{Re}\, s = \varkappa$.

Theorem 2.1 *([12]). Let $n \geq 2$, $1 < p < \infty$, $\varkappa = (\lambda+n)/p$ and let the following assumptions be satisfied*

I) there exist numbers $N_0 \in \mathbf{Z}^0_+$, $C_1 > 0$ and $C_2 > 0$ such that

$$\left|\xi^k \frac{\partial^k}{\partial \xi^k}\mu_m\left(\frac{\lambda+n}{p} + i\xi\right)\right| \leq C_1, \qquad \xi \in \mathbf{R}\setminus\{0\},$$

where $m = 0, 1, \ldots, N_0$ and $k = 0, 1$ with C_1 not depending on ξ and m;

II) there exists a function $M(\xi,\eta) : \mathbf{R}\times[N_0,\infty) \to \mathbf{C}$ such that $M(\xi,m) = \mu_m((\lambda+n)/p + i\xi)$ for all $m \geq \varkappa$ and

$$\left|\xi^k\eta^l \frac{\partial^k}{\partial\xi^k}\frac{\partial^l}{\partial\eta^l}M(\xi,\eta)\right| \leq C_2, \qquad \xi \in \mathbf{R}\setminus\{0\}, \eta \geq N_0,$$

$l = 0, 1, \ldots, [(n+1)/2]$, $k = 0, 1$, where C_2 does not depend on ξ and η.

Then $\|A_\varkappa\varphi\|_{L_p(\mathbf{R}^n,|x|^\lambda)} \leq b(C_1+C_2)\|\varphi\|_{L_p(\mathbf{R}^n,|x|^\lambda)}$, $\varphi \in C^\infty_{0,0}$, where b is some absolute constant not depending on $\{\mu_m(s)\}_{m=0}^\infty$.

4°. On the analyticity of an integral depending on a parameter. The following lemma is known.

Lemma 2.3 *([16], Subsection 5.2 of Ch.1). Let $f(x,z)$ be an analytic function with respect to $z \in D \subset \mathbf{C}$ for almost all $x \in \Omega \subseteq \mathbf{R}^n$ which is dominated by a function integrable in x not depending on z: $|f(x,z)| \leq F(x) \in L_1(\Omega)$. Then the integral $\int_\Omega f(x,z)\,dx$ is an analytic function in D.*

5°. L_p-mapping properties of operators with homogeneous kernels. The following statement is contained in Theorem 1 from [6].

Theorem 2.2 *Let*

$$(K\varphi)(x) = \int_{\mathbf{R}^n} k\left(\frac{|x|}{|y|}, x'\cdot y'\right)\varphi(y)\frac{dy}{|y|^n}$$

and let the condition

$$A = |S_{n-2}|\int_0^\infty\int_{-1}^1 |k(\rho,t)|\,\rho^{(n+\lambda)/p-1}(1-t^2)^{(n-3)/2}\,d\rho\,dt < \infty \tag{2.1}$$

be satisfied. Then K is bounded in $L_p(\mathbf{R}^n,|x|^\lambda)$, $1 < p < \infty$, and $\|K\| \leq A$.

3 Fractional powers $(-|x|^2\Delta)^{-\alpha/2}$, $\mathtt{Re}\,\alpha > 0$, in L_p-spaces

1°. Heuristic arguments. Applying the inverse Mellin transform in the equality

$$(\mathfrak{M}(-|x|^2\Delta\varphi)_{m\mu})(s) = (m+n-2-s)(m+s)(\mathfrak{M}\varphi_{m\mu})(s), \qquad \varphi \in C^\infty_{0,0}, \tag{3.1}$$

we have

$$(-|x|^2\Delta\varphi)_{m\mu}(|x|) = \frac{1}{2\pi i}\int_{\mathtt{Re}\,s=\varkappa} |x|^{-s}(m+n-2-s)(m+s)(\mathfrak{M}\varphi_{m\mu})(s)\,ds.$$

Expanding the function $-|x|^2(\Delta\varphi)(x)$ into the Fourier-Laplace series (see [16], p.34), we arrive at the equality

$$-|x|^2(\Delta\varphi)(x) = \sum_{m=0}^{\infty}\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')(-|x|^2\Delta\varphi)_{m\mu}(|x|) =$$

$$= \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')\int_{\mathtt{Re}\,s=\varkappa} |x|^{-s}(m+n-2-s)(m+s)(\mathfrak{M}\varphi_{m\mu})(s)\,ds,$$

where $\varphi \in C^\infty_{0,0}$, $\varkappa \in \mathbf{R}$.

Basing on this equality, it is natural to define negative powers $(-|x|^2\Delta)^{-\alpha/2}\varphi = \mathbf{I}^\alpha_\varkappa\varphi$ with $\mathtt{Re}\,\alpha > 0$ and $\varkappa \in \mathbf{R}$, by (1.3).

2°. Mapping properties of the operator $\mathbf{I}^\alpha_\varkappa$ in L_p. The following theorem provides L_p-estimates for the operator $\mathbf{I}^\alpha_\varkappa$ defined in (1.3).

Theorem 3.1 *Let $n \geq 2$, $1 < p < \infty$, $\varkappa = n/p$, $\mathtt{Re}\,\alpha > 0$. Then*

$$\|\mathbf{I}^\alpha_\varkappa\varphi\|_p \leq C\|\varphi\|_p$$

for $\varphi \in C^\infty_{0,0}$ where C does not depend on φ.

The statement of Theorem 3.1 follows from Theorem 2.1, when $p \neq 2$ and $n = 2$, or $p \notin \{n/(n-1), n/(n-2)\}$ and $n \geq 3$. The proof being direct, we omit it. In the remaining cases, the proof is derived from the Riesz interpolation theorem.

3°. Integral representation for the operator $\mathbf{I}^\alpha_\varkappa$, $n \geq 3$. We introduce the kernel

$$a^\alpha_m(\rho) = \frac{(2m+n-2)^{\frac{1-\alpha}{2}}}{\sqrt{\pi}\Gamma(\alpha/2)}|\ln\rho|^{\frac{\alpha-1}{2}}K_{\frac{\alpha-1}{2}}\left(\frac{2m+n-2}{2}|\ln\rho|\right)\rho^{\frac{2-n}{2}}, \quad 0<\rho<\infty$$

where $K_\nu(z)$ is the McDonald function of order ν. Its Mellin transform is given by

$$(\mathfrak{M}a^\alpha_m)(s) = (m+n-2-s)^{-\alpha/2}(m+s)^{-\alpha/2}, \qquad -m < \mathtt{Re}\,s < m+n-2, \tag{3.2}$$

which is verified by means of the formula

$$\int_0^\infty t^\nu K_\nu(at)cosh\,ptdt = \frac{2^{\nu-1}\sqrt{\pi}\Gamma\left(\nu+\frac{1}{2}\right)a^\nu}{(a^2-p^2)^{\nu+\frac{1}{2}}}, \tag{3.3}$$

$$a > 0, \quad \nu > -\frac{1}{2}, \qquad |p| < a$$

(see [11], formula 2.16.6.6; it may be also obtained from the formula more known formula 6.621.3 in [5] by use of some standard properties of the Gauss hypergeometric function).

Lemma 3.1 *Let* $\mathtt{Re}\,\alpha > 0$, $n \geq 3$, $m \in \mathbf{Z}^0_+$ *and* $\varkappa \in (0, n-2)$. *Then the representation*

$$\frac{1}{2\pi i}\int_{\mathtt{Re}\,s=\varkappa} \rho^{-s}(m+n-2-s)^{-\alpha/2}(m+s)^{-\alpha/2}(\mathfrak{M}\psi)(s)\,ds =$$
$$= \int_0^\infty a^\alpha_m\left(\frac{\rho}{r}\right)\psi(r)\frac{dr}{r}, \qquad \psi \in C^\infty_{0,0}, \tag{3.4}$$

is valid.

Proof. Denote $h(g) = \displaystyle\int_0^\infty a^\alpha_m\left(\frac{\rho}{r}\right)\psi(r)\frac{dr}{r}$. Using the well-known properties of the McDonald function (see, for example, [3]), it is easy to verify that $\rho^{\varkappa-1}a^\alpha_m(\rho) \in L^c(0,\infty)$, if $\varkappa \in (0, n-2)$. Hence, $t^{\varkappa-1}h(t) \in L^c(0,\infty)$ and $(\mathfrak{M}h)(s) = (\mathfrak{M}\psi)(s)(\mathfrak{M}a^\alpha_m)(s)$ by Lemma 2.2. Then Lemma 2.1 and relation (3.2) yield (3.4). ∎

Making use of Lemma 3.1 in (1.3), we have

$$(\mathbf{I}^\alpha_\varkappa\varphi)(x) = \sum_{m=0}^\infty\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')\int_0^\infty a^\alpha_m\left(\frac{|x|}{\rho}\right)\varphi_{m\mu}(\rho)\frac{d\rho}{\rho},$$

$0 < \varkappa < n-2$, $\varphi \in C^\infty_{0,0}$. Taking into account the equality

$$\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')\mathcal{Y}_{m\mu}(\sigma) = \frac{d_n(m)}{|S_{n-1}|}P_m(x'\cdot\sigma), \tag{3.5}$$

(see [16], formula 1.57), where the Legendre polynomial

$$P_m(\theta) = \frac{(n-3)!m!}{(m+n-3)!}\mathcal{C}^{\frac{n-2}{2}}_m(\theta), \qquad -1 \leq \theta \leq 1, \tag{3.6}$$

is defined via the Gegenbauer polynomial $C^\lambda_m(\theta)$, we obtain

$$(\mathbf{I}^\alpha_\varkappa\varphi)(x) = \frac{1}{|S_{n-1}|}\sum_{m=0}^\infty d_n(m)\int_0^\infty\int_{S_{n-1}} \varphi(r\sigma)a^\alpha_m\left(\frac{|x|}{r}\right)P_m(x'\cdot\sigma)\frac{dr}{r}\,d\sigma$$
$$= \frac{1}{|S_{n-1}|}\sum_{m=0}^\infty d_n(m)\int_{\mathbf{R}^n}\varphi(y)a^\alpha_m\left(\frac{|x|}{|y|}\right)P_m(x'\cdot y')\frac{dy}{|y|^n}, \tag{3.7}$$

$\varphi \in C^\infty_{0,0}$, $0 < \varkappa < n-2$. To represent the potential $\mathbf{I}^\alpha_\varkappa\varphi$ as a radial-spherical convolution, we consider the kernel

$$k_\alpha(\rho,\theta) = \frac{1}{|S_{n-1}|}\sum_{m=0}^\infty d_n(m)a^\alpha_m(\rho)P_m(\theta),$$

where $\mathtt{Re}\,\alpha > 0$, $0 < \rho < \infty$, and $-1 \leq \theta \leq 1$.

Lemma 3.2 *Let* $n \geq 3$, $\mathtt{Re}\,\alpha > 0$, $-1 \leq \theta \leq 1$, $\rho > 0$ *and* $\rho \neq 1$. *Then*

$$k_\alpha(\rho,\theta) = \frac{|\ln\rho|^{\alpha-1}\rho^{\frac{2-n}{2}}}{|S_{n-1}|2^{\alpha-1}\Gamma(\frac{\alpha}{2})}\int_1^\infty (t^2-1)^{\frac{\alpha}{2}-1}\frac{|\rho^{\frac{n-2}{2}t}-\rho^{\frac{n+2}{2}t}|}{(1-2\theta\rho^t+\rho^{2t})^{\frac{n}{2}}}\,dt = \tag{3.8}$$

$$= \frac{2^{1-\alpha}\rho^{\frac{2-n}{2}}}{|S_{n-1}|\Gamma\left(\frac{\alpha}{2}\right)} \int_0^{\widetilde{\rho}} \left(\ln^2 s - \ln^2 \rho\right)^{\frac{\alpha}{2}-1} \frac{s^{\frac{n-4}{2}}(1-s^2)\,ds}{(1-2\theta s+s^2)^{\frac{n}{2}}}\ ,$$

where $\widetilde{\rho} - \min\left(\rho, \frac{1}{\rho}\right) = \rho^{sign\,(1-\rho)}$.

Proof. We rewrite $k_\alpha(\rho,\theta)$ as:

$$k_\alpha(\rho,\theta) = \frac{1}{|S_{n-1}|(n-2)} \sum_{m=0}^{\infty} (2m+n-2) a_m^\alpha(\rho) \mathcal{C}_m^{\frac{n-2}{2}}(\theta). \tag{3.9}$$

Using the formula 8.432(3) from [5], we obtain

$$a_m^\alpha(\rho) = \frac{|\ln\rho|^{\alpha-1}\rho^{\frac{2-n}{2}}}{\Gamma^2(\frac{\alpha}{2})2^{\alpha-1}} \int_1^\infty (t^2-1)^{\frac{\alpha}{2}-1} \exp\left(-(m+\frac{n-2}{2})|\ln\rho|t\right)\,dt,$$

whence

$$k_\alpha(\rho,\theta) = \frac{|\ln\rho|^{\alpha-1}\rho^{\frac{2-n}{2}}2^{1-\alpha}}{|S_{n-1}|(n-2)\Gamma^2(\frac{\alpha}{2})} \sum_{m=0}^{\infty}(2m+n-2)\mathcal{C}_m^{\frac{n-2}{2}}(\theta) \times$$
$$\times \int_1^\infty (t^2-1)^{\frac{\alpha}{2}-1} \exp\left(-(m+\frac{n-2}{2})|\ln\rho|t\right)\,dt$$

by (3.9). Changing the order of integration and summation, we arrive at the equality

$$k_\alpha(\rho,\theta) = \frac{|\ln\rho|^{\alpha-1}\rho^{\frac{2-n}{2}}}{|S_{n-1}|2^{\alpha-1}\Gamma^2(\frac{\alpha}{2})} \int_1^\infty (t^2-1)^{\frac{\alpha}{2}-1} S_n(\widetilde{\rho}^t;\theta)\,dt,$$

where

$$S_n(z;\theta) = \frac{1}{n-2}\sum_{m=0}^{\infty}(2m+n-2)z^{m+\frac{n-2}{2}}\mathcal{C}_m^{\frac{n-2}{2}}(\theta),$$

$0<z<1$, $-1\le\theta\le1$. We have

$$S_n(z;\theta) = \frac{2}{n-2} z\frac{d}{dz}\left(z^{\frac{n-2}{2}}\sum_{m=0}^{\infty} z^m \mathcal{C}_m^{\frac{n-2}{2}}(\theta)\right). \tag{3.10}$$

Since $(1-2rt+r^2)^{-\lambda} = \sum_{m=0}^\infty \mathcal{C}_m^\lambda(t) r^m$ (see [16], formula (1.16)), we obtain

$$S_n(z;\theta) = \frac{2}{n-2} z\frac{d}{dz}\left(\frac{z}{1-2\theta z+z^2}\right)^{\frac{n-2}{2}} = \frac{z^{\frac{n-2}{2}} - z^{\frac{n+2}{2}}}{(1-2\theta z+z^2)^{\frac{n}{2}}}$$

which yields (3.8). ■

We also need the following estimates for the kernel $k_\alpha(\rho,\theta)$.

Lemma 3.3 *Let* $\mathtt{Re}\,\alpha>0$, $n\ge3$, $0<\rho<\infty$, $-1\le\theta\le1$.

I) If $-1 \le \theta \le 1/2$, $0 < \rho < \infty$, then

$$|k_\alpha(\rho,\theta)| \le C \begin{cases} |\ln\rho|^{\mathtt{Re}\,\alpha/2-1}, & 0<\rho<1/2; \\ |\ln\rho|^{\mathtt{Re}\,\alpha-1}, & 1/2<\rho<2, 0<\mathtt{Re}\,\alpha<1; \\ |\ln|\ln\rho||, & 1/2<\rho<2, \mathtt{Re}\,\alpha=1; \\ 1, & 1/2<\rho<2, \mathtt{Re}\,\alpha>1; \\ |\ln\rho|^{\mathtt{Re}\,\alpha/2-1}\rho^{2-n}, & 2<\rho<\infty. \end{cases} \tag{3.11}$$

II) If $1/2 < \theta \le 1$, $0 < \widetilde{\rho} \equiv \min\{\rho, 1/\rho\} < \theta - 1/4$, then

$$|k_\alpha(\rho,\theta)| \le C|\ln\rho|^{\mathtt{Re}\,\alpha/2-1} \begin{cases} 1, & 0<\rho<1/2; \\ \rho^{2-n}, & 2<\rho<\infty. \end{cases} \tag{3.12}$$

III) Let $1/2 < \theta \le 1$, $\theta - 1/4 \le \widetilde{\rho} < 1$. The following inequalities are valid:
a) if $0 < \mathtt{Re}\,\alpha < 2n-2$, then

$$|k_\alpha(\rho,\theta)| \le C\left[|\rho-1|^{(\mathtt{Re}\,\alpha/2-1)_-}(1-2\theta\widetilde{\rho}+\widetilde{\rho}^2)^{(\mathtt{Re}\,\alpha-2n+2)/4}\right. \tag{3.13}$$

$$\left.+|\rho-1|^{1+(\mathtt{Re}\,\alpha/2-1)_-}(1-2\theta\widetilde{\rho}+\widetilde{\rho}^2)^{(\mathtt{Re}\,\alpha-2n)/4}+1\right] \equiv Cu_{\mathtt{Re}\,\alpha}(\rho;\theta);$$

b) if $\mathtt{Re}\,\alpha > 0$ then

$$|k_\alpha(\rho,\theta)| \le C|\rho-1|^{(\mathtt{Re}\,\alpha/2-1)_-+(\mathtt{Re}\,\alpha/2-n+1)_-} \equiv Cv_{\mathtt{Re}\,\alpha}(\rho;\theta). \tag{3.14}$$

Moreover, the constant C can be chosen independent of α in a small neighborhood of each point α of the half-plane $\mathtt{Re}\,\alpha > 0$.

The proof of lemma is reduced to the direct estimation of the right-hand side of (3.8).

Let

$$(\mathbf{I}^\alpha\varphi)(x) = \int_{\mathbf{R}^n} \varphi(y) k_\alpha\left(\frac{|x|}{|y|}, x'\cdot y'\right)\frac{dy}{|y|^n}\ . \tag{3.15}$$

The following theorem provides the representation for the operator (1.3) in the form of a radial-spherical convolution.

Theorem 3.2 *Let $\mathtt{Re}\,\alpha > 0$, $n \ge 3$, $0 < \varkappa < n-2$, $\varphi \in C^\infty_{0,0}$. Then*

$$(\mathbf{I}^\alpha_\varkappa\varphi)(x) \equiv (\mathbf{I}^\alpha\varphi)(x), \tag{3.16}$$

where $(\mathbf{I}^\alpha_\varkappa\varphi)(x)$ was defined in (1.3).

Proof. Let $\mathtt{Re}\,\alpha > n-1$ first. Changing the order of integration and summation in (3.7), we obtain (3.16). This interchange is justified by the inequality

$$\sum_{m=1}^\infty d_n(m)\int_{\mathbf{R}^n}\left|\varphi(y)a^\alpha_m\left(\frac{|x|}{|y|}\right)P_m(x'\cdot y')\right|\frac{dy}{|y|^n} < \infty. \tag{3.17}$$

The last inequality follows from the estimates

$$|P_m(\theta)| \le 1, \qquad -1 \le \theta \le 1, \tag{3.18}$$

$$d_n(m) \le Cm^{n-2}, \tag{3.19}$$

(see [16], inequality (1.55) and Corollary to Lemma 1.4) and the equality

$$\int_{\mathbf{R}^n} a_m^{\mathtt{Re}\,\alpha}\left(\frac{1}{|y|}\right)\frac{dy}{|y|^n} = (\mathfrak{M}a_m^{\mathtt{Re}\,\alpha})(0).$$

The validity of the equality in (3.16) for $\mathtt{Re}\,\alpha > 0$ isderived from the analyticity of its both sides in the half-plane $\mathtt{Re}\,\alpha > 0$. The analiticity of the left-hand side is derived from the uniform convergence of the series (1.3) in the half-plane $\mathtt{Re}\,\alpha > 0$ (for any fixed $x \neq 0$). We omit the proof of this fact, because it is purely technical. Let us prove the analyticity of the right-hand side of (3.16). Analyticity of the function $k_\alpha(\rho,\theta)$ defined in (3.8) in the half-plane $\mathtt{Re}\,\alpha > 0$ follows from Lemma 2.3 in view of the estimate

$$|(t^2-1)^{\alpha/2-1}| \leq (t^2-1)^{\varepsilon/2-1} + (t^2-1)^{N/2-1},$$

where $t > 1$, $0 < \varepsilon < \mathtt{Re}\,\alpha < N < \infty$. Then the application of Lemma 2.3 provides the analyticity of the integral $\mathbf{I}^\alpha\varphi$, defined in (3.15), in the half-plane $\mathtt{Re}\,\alpha > 0$ by virtue of the estimate

$$|k_\alpha(\rho,\theta)| \leq C \begin{cases} u_\varepsilon(\rho,\theta) + u_N(\rho,\theta), & 0 < \varepsilon < \mathtt{Re}\,\alpha < N = 2n-2, \\ v_\delta(\rho,\theta) + u_L(\rho,\theta), & 2n-4 < \delta < \mathtt{Re}\,\alpha < L < \infty, \end{cases} \tag{3.20}$$

where $1/2 < \theta \leq 1$ and $\theta - 1/4 \leq \widetilde{\rho} \equiv \min\{\rho, 1/\rho\} < 1$ and the functions u_γ and v_γ were defined in (3.13) and (3.14), the constant C in (3.20) not depending on α in a small neighborhood of each point α, $\mathtt{Re}\,\alpha > 0$. The integrability of the function in the right-hand side of (3.20) - which is required by Lemma 2.3 - follows from the relations:

$$u_\gamma\left(\frac{|x|}{|y|}, x'\cdot y'\right)\xi(x'\cdot y') \in L_1\left(\{y \in \mathbf{R}^n : \frac{1}{2} < \frac{|x|}{|y|} < 2\}\right),$$

if $\gamma \in (0, 2n-2]$, where

$$\xi(\theta) = \begin{cases} 1, & \text{if } 1/2 < \theta \leq 1, \\ 0, & \text{if } -1 \leq \theta \leq 1/2, \end{cases}$$

and

$$v_\gamma\left(\frac{|x|}{|y|}, x'\cdot y'\right)\xi(x'\cdot y') \in L_1\left(\{y \in \mathbf{R}^n : \frac{1}{2} < \frac{|x|}{|y|} < 2\}\right),$$

if $\gamma > 2n-4$. ■

4°. Representation of the potential $\mathbf{I}^\alpha_\varkappa\varphi$, $\varphi \in L_p$. To extend (3.16) to the whole space L_p, we need the following theorem.

Theorem 3.3 *Let $\mathtt{Re}\,\alpha > 0$, $n \geq 3$ and $n/(n-2) < p < \infty$. Then the operator $\mathbf{I}^\alpha$ is bounded in L_p.*

The statement of Theorem 3.3 follows from Theorem 2.2. The latter is applicable in view of the inequality

$$A \leq C\left(\int_0^{1/2} |\ln\rho|^{\mathtt{Re}\,\alpha/2-1}\rho^{n/p-1}d\rho + \right.$$
$$\left. + \int_2^\infty |\ln\rho|^{\mathtt{Re}\,\alpha/2-1}\rho^{n/p+1-n}d\rho + 1\right) =$$
$$= C\left(\int_{\ln 2}^\infty t^{\mathtt{Re}\,\alpha/2-1}\left[\exp(-nt/p) + \exp((n/p+2-n)t)\right]dt + 1\right) < \infty.$$

Since the operators on both sides of the relation (3.16) are bounded in L_p , if $\mathrm{Re}\,\alpha > 0$, $n \geq 3$ and $n/(n-2) < p < \infty$, this relation is extended by continuity to the whole L_p under the given restrictions on the parameters. ■

4 Representation of the potential $\mathbf{I}^\alpha\varphi$ via the Riemann-Liouville fractional integral of a strongly continuous semi-group

In this section we study the semi-group T_t generated by the infinitesimal operator $|x|^2\Delta$. Our main goal is to prove the representation (1.5).

1°. Heuristic arguments.

The semi-group T_t, $t > 0$, generated by the infinitesimal operator $|x|^2\Delta$ is

$$(T_t\varphi)(x) = \exp(t|x|^2\Delta)\varphi(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!}(|x|^2\Delta)^k\varphi(x), \quad \varphi \in C_{0,0}^\infty.$$

By virtue of (3.1) we obtain

$$\begin{aligned}(\mathfrak{M}(T_t\varphi)_{m\mu})(s) &= \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\mathfrak{M}\left((|x|^2\Delta)^k\varphi(x)\right)_{m\mu}\right](s) = \\ &= \sum_{k=0}^{\infty} \frac{t^k(s+m)^k(s-m-n+2)^k}{k!}(\mathfrak{M}\varphi_{m\mu})(s) = \\ &= \exp\left((s+m)(s-m-n+2)t\right)(\mathfrak{M}\varphi_{m\mu})(s),\end{aligned}$$

whence

$$(T_t\varphi)_{m\mu}(\rho) = \frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\varkappa} |x|^{-s}\exp\left((s+m)(s-m-n+2)t\right)(\mathfrak{M}\varphi_{m\mu})(s)\,ds.$$

Expanding $(T_t\varphi)(x)$ into the Fourier-Laplace series, we have

$$\begin{aligned}(T_t\varphi)(x) = \frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')\int_{\mathrm{Re}\, s=\varkappa} |x|^{-s} \times \\ \times\exp\left((s+m)(s-m-n+2)t\right)(\mathfrak{M}\varphi_{m\mu})(s)\,ds,\end{aligned} \tag{4.1}$$

where $\varphi \in C_{0,0}^\infty$, $\varkappa \in \mathbf{R}$.

2°. Mapping properties of the operator T_t in L_p. The following theorem provides the L_p -estimates for the operator $T_t, t > 0$, defined in (4.1).

Theorem 4.1 *Let $n \geq 2$, $1 < p < \infty$, $0 < t < M$ and $\varkappa = n/p$. Then $\|T_t\varphi\|_p \leq C\|\varphi\|_p$, $\varphi \in C_{0,0}^\infty$, where the constant $C = C(M)$ does not depend on t and φ.*

The proof is reduced to the direct application of Theorem 2.1.

3°. Integral representation for the operator T_t. Here we obtain a representation for the operator T_t, defined in (4.1), in the form of a radial-spherical convolution. We denote

$$b_m^t(\rho) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(\ln\rho)^2}{4t} - \frac{(2m+n-2)^2}{4}t\right)\rho^{(2-n)/2}, \qquad n \geq 2,$$

where $0 < \rho < \infty$ and $t > 0$ and use the following technical lemma.

Lemma 4.1 *Let $t > 0$, $n \geq 2$, $m \in \mathbf{Z}_+^0$, $\varkappa \in \mathbf{R}$. Then*

$$\frac{1}{2\pi i}\int_{\mathrm{Re}\, s=\varkappa} \rho^{-s} \exp\left((s+m)(s-m-n+2)t\right)(\mathfrak{M}\psi)(s)\, ds =$$
$$= \int_0^\infty b_m^t\left(\frac{\rho}{r}\right)\psi(r)\frac{dr}{r}, \qquad \psi \in C_{0,0}^\infty. \tag{4.2}$$

Proof. We observe that

$$(\mathfrak{M}b_m^t)(s) = \exp\left((s+m)(s-m-n+2)t\right), \qquad -\infty < \mathrm{Re}\, s < \infty, \tag{4.3}$$

which is a consequence of the formula 2.3.16(11) from [10]. To prove (4.2), we denote

$$H_t(\rho) = \int_0^\infty b_m^t\left(\frac{\rho}{r}\right)\psi(r)\frac{dr}{r}.$$

Since $r^{\varkappa-1}b_m^t(r)$ and $r^{\varkappa-1}\psi(r)$ are in $L^c(0,\infty)$ for any $\varkappa \in \mathbf{R}$, by Lemma 3.1 we obtain that $r^{\varkappa-1}H_t(\rho) \in L^c(0,\infty)$ and $(\mathfrak{M}H_t)(s) = (\mathfrak{M}b_m^t)(s)(\mathfrak{M}\psi)(s)$. Making use of Lemma 2.1 and equality (4.3), we arrive at (4.2). ■

Theorem 4.2 *Let $n \geq 2$, $t > 0$, $\varkappa \in \mathbf{R}$ and $\varphi \in C_{0,0}^\infty$. Then*

$$(T_t\varphi)(x) = \int_{\mathbf{R}^n} \varphi(y) h_t\left(\frac{|x|}{|y|}\right) S_t(x' \cdot y')\frac{dy}{|y|^n}, \tag{4.4}$$

where

$$h_t(\rho) = b_0^t(\rho) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{(\ln\rho)^2}{4t} - \frac{(n-2)^2}{4}t\right)\rho^{(2-n)/2} \tag{4.5}$$

and

$$S_t(\theta) = \frac{1}{|S_{n-1}|}\sum_{m=0}^\infty d_n(m)\exp\left(-m(m+n-2)t\right)P_m(\theta). \tag{4.6}$$

Proof. From (4.1), in view of (4.2) we have

$$(T_t\varphi)(x) = \sum_{m=0}^\infty \sum_{\mu=1}^{d_n(m)} \mathcal{Y}_{m\mu}(x')\int_0^\infty b_m^t\left(\frac{|x|}{r}\right)\int_{S_{n-1}} \varphi(r\sigma)\mathcal{Y}_{m\mu}(\sigma)\, d\sigma\frac{dr}{r}, \tag{4.7}$$

whence by virtue of (3.5) we obtain

$$(T_t\varphi)(x) = \frac{1}{|S_{n-1}|}\sum_{m=0}^{\infty} d_n(m)\int_0^{\infty}\int_{S_{n-1}} \varphi(r\sigma) b_m^t\left(\frac{|x|}{r}\right) P_m(x'\cdot\sigma)\frac{dr}{r}\,d\sigma =$$

$$= \frac{1}{|S_{n-1}|}\sum_{m=0}^{\infty} d_n(m)\int_{\mathbf{R}^n} \varphi(y) b_m^t\left(\frac{|x|}{|y|}\right) P_m(x'\cdot y')\frac{dy}{|y|^n}. \tag{4.8}$$

Changing the order of integration and summation in (4.8), we obtain (4.4). This interchange is justified by the fact that

$$A(x) :\equiv \sum_{m=1}^{\infty} d_n(m)\int_{\mathbf{R}^n}\left|\varphi(y) b_m^t\left(\frac{|x|}{|y|}\right) P_m(x'\cdot y')\right|\frac{dy}{|y|^n} < \infty.$$

Indeed, the application of (3.15), (3.16) and (4.3) yields

$$\begin{aligned} A(x) &\le C\sum_{m=1}^{\infty} m^{n-2}\int_{\mathbf{R}^n} b_m^t\left(\frac{|x|}{|y|}\right)\frac{dy}{|y|^n} \\ &= C\sum_{m=1}^{\infty} m^{n-2}\int_{\mathbf{R}^n} b_m^t\left(\frac{1}{|y|}\right)\frac{dy}{|y|^n} = C\sum_{m=1}^{\infty} m^{n-2}(\mathfrak{M}b_m^t)(0) \\ &= C\sum_{m=1}^{\infty} m^{n-2}\exp\left(-m(m+n-2)t\right) < \infty. \end{aligned}$$

■

The following theorem plays an important role in the justification of the inversion formula for the potential $\mathbf{I}^\alpha\varphi$.

Theorem 4.3 *Let $n \ge 2$, $t > 0$ and $1 < p < \infty$. Then*

$$\|T_t\varphi\|_p \le C_t\|\varphi\|_p, \qquad \varphi \in L_p, \tag{4.9}$$

where

$$C_t = C(1+t^{1-n})\exp\left[\frac{n}{p}\left(2-n+\frac{n}{p}\right)t\right],$$

with the constant C not depending on t.

Proof. The proof is obtained by direct application of Theorem 2.2 to the right-hand side of (4.6). Let us show that $A \le C_t$, where A is the constant (2.1). The application of (3.15) and (3.19) yields

$$A = \frac{\exp(-(n-2)^2t/4)}{2\sqrt{\pi t}|S_{n-1}|}\int_0^{\infty}\exp\left(-\frac{(\ln\rho)^2}{4t}\right)\rho^{(2-n)/2+n/p-1}d\rho\,\times$$

$$\times\int_{S_{n-1}}\left(\sum_{m=0}^{\infty} d_n(m)\exp\left(-m(m+n-2)t\right)|P_m(\theta\cdot\sigma|\right)d\theta \le$$

$$C\frac{\exp(-(n-2)^2t/4)}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\exp\left(-\frac{x^2}{4t}+\left(\frac{2-n}{2}+\frac{n}{p}\right)x\right)dx\,\times$$

$$\times\left[1+\sum_{m=1}^{\infty} m^{n-2}\exp\left(-m(m+n-2)t\right)\right].$$

Taking into account the formula 2.3.16(11) from [10] and the estimate

$$x^c e^{-ctx} \le (et)^{-c},$$

where $t > 0$, $c > 0$, and $x > 0$, we have

$$A \le C \exp\left[\frac{n}{p}\left(2 - n + \frac{n}{p}\right)t\right] \times$$
$$\times \left(1 + \sum_{m=1}^{\infty}\left[m^{n-2}\exp\left(-(n-2)tm\right)\exp(-m^2 t)\right]\right) \le$$
$$\le C \exp\left[\frac{n}{p}\left(2 - n + \frac{n}{p}\right)t\right]\left(1 + t^{2-n}\sum_{m=1}^{\infty}\exp(-mt)\right).$$

In view of the relation

$$1 + t^{2-n}\sum_{m=1}^{\infty}\exp(-mt) = 1 + \frac{t^{2-n}}{\exp(t) - 1} \le 1 + t^{1-n}$$

we obtain (4.9). ■

We note that the above theorem provides sharper estimates for the norms $\|T_t\|_{L_p \to L_p}$ in comparison with those obtained in Theorem 4.1. It states that

$$\|T_t\|_{L_p \to L_p} \le C \exp\left[\frac{n}{p}\left(2 - n + \frac{n}{p}\right)t\right], \qquad t > 0,\ 1 < p < \infty, \tag{4.10}$$

where C does not depend on t.

We also give the following point-wise estimate for $(T_t\varphi)(x)$, which will be used in the justification of the inversion formula for the potential $\mathbf{I}^\alpha\varphi$.

Lemma 4.2 *Let $\varphi \in C_{0,0}^{\infty}$ and $n \ge 2$. Then*

$$|(T_t\varphi)(x)| \le C|x|^{-1/r'}t^{-1/(2r)}\exp\left\{\frac{1}{r}\left(2 - n + \frac{1}{r}\right)t\right\}, \quad t > 0,$$

where $1 < r < \infty$, $x \in \mathbf{R}^n \setminus \{0\}$ and C depends only on φ, n and r.

The proof is based on the equality (4.7).

4°. Further properties of the operator T_t. Theorems 4.4 and 4.5 below show that the family $\{T_t\}_{t\ge 0}$ is a semi-group strongly continuous in L_p.

Theorem 4.4 *Let $n \ge 2$, $1 < p < \infty$ and $\varphi \in L_p(\mathbf{R}^n)$. Then*

$$(T_t T_\tau \varphi)(x) = (T_{\tau+t}\varphi)(x), \qquad t > 0, \tau > 0. \tag{4.11}$$

Proof. Since the operators on both sides of (4.11) are bounded in L_p, it remains to verify this equality for $\varphi \in C^\infty_{0,0}$. For such $\varphi(x)$ we have by (4.4)

$$(T_t T_\tau \varphi)(x) = \int_{\mathbf{R}^n} h_t\left(\frac{|x|}{|y|}\right) S_t(x' \cdot y') \left(\int_{\mathbf{R}^n} h_\tau\left(\frac{|y|}{|z|}\right) S_\tau(y' \cdot z')\varphi(z)\frac{dz}{|z|^n}\right)\frac{dy}{|y|^n}$$
$$\int_{\mathbf{R}^n} \varphi(z)\left(\int_0^\infty h_t\left(\frac{|x|}{\rho}\right) h_\tau\left(\frac{\rho}{|z|}\right)\frac{d\rho}{\rho}\right)\left(\int_{S_{n-1}} S_t(x'\cdot\sigma)S_\tau(\sigma\cdot z')\,d\sigma\right)\frac{dz}{|z|^n}.$$

Application of the equalities

$$\int_0^\infty h_t(a/\rho)h_\tau(\rho/b)\frac{d\rho}{\rho} = h_{t+\tau}(a/b), \qquad t>0, \tau>0, a>0, b>0,$$

and

$$\int_{S_{n-1}} S_t(x'\cdot\sigma)S_\tau(\sigma\cdot y')\,d\sigma = S_{t+\tau}(x', y'),$$

obtained by simple calculations, yields (4.11). ■

Theorem 4.5 *Let $n \geq 2$ and $\varphi \in L_p(\mathbf{R}^n)$, $1 < p < \infty$. Then*

$$\lim_{t\to 0} \|T_t\varphi - \varphi\|_p = 0 \tag{4.12}$$

The verification of (4.12) is now easily obtained by means of the Banach-Steinhaus theorem: the uniform estimate

$$\sup_{0<t<1} \|T_t\|_{L_p\to L_p} < \infty$$

is a consequence of (4.9), while the proof of (4.12) for $\varphi \in C^\infty_{0,0}$ is direct.

The following theorem plays a crucial role in the justification of the inversion formula for the operator $\mathbf{I}^\alpha$ within the framework of L_p-spaces.

Theorem 4.6 *Let $n \geq 3$, $\mathtt{Re}\,\alpha > 0$. On functions $\varphi \in C^\infty_{0,0}$ the operator (3.15) may be represented as*

$$(\mathbf{I}^\alpha\varphi)(x) = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty t^{\alpha/2-1}(T_t\varphi)(x)\,dt. \tag{4.13}$$

Proof. Let $\mathtt{Re}\,\alpha > 2n-2$. Substituting (4.4) into the integral on the right-hand side of (4.13) and changing the order of integration, after simple calculations we arrive at the left-hand side of (4.13).

The validity of (4.13) in the case $\mathtt{Re}\,\alpha > 0$ follows from the analyticity of both sides in the half-plane $\mathtt{Re}\,\alpha > 0$. The analyticity of the left-hand side follows from the uniform convergence of the series (1.3) in the half-plane $\mathtt{Re}\,\alpha > 0$ (as was mentioned in the proof of Theorem 3.2). The analyticity of the right-hand side is evident in view of Lemma 3.2. ■

5 Inversion of the potential $f = \mathbf{I}^\alpha \varphi$, $\varphi \in L_p$

1°. The case $\alpha > 0$. Denote

$$(\mathbf{D}^\alpha_\varepsilon f)(x) = \frac{1}{\varkappa(\alpha/2, l)} \int_\varepsilon^\infty (E - T_t)^l f(x) \frac{dt}{t^{\alpha/2+1}}, \qquad \alpha > 0,$$

where

$$\begin{aligned} (E - T_t)^l f(x) &= \sum_{k=0}^{l} (-1)^k \binom{l}{k} (T_{kt} f)(x), \\ \varkappa(\lambda, l) &= \int_0^\infty \frac{(1 - e^{-t})^l}{t^{1+\lambda}} \, dt, \qquad l > \lambda. \end{aligned} \tag{5.1}$$

Let

$$(J^\alpha g)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty g(\tau)(\tau - t)^{\alpha-1} d\tau, \qquad \alpha > 0$$

be the Liouville fractional integral.

Theorem 5.1 *Let $n \geq 3$, $\alpha > 0$, and $\varphi \in L_p$ with $n/(n-2) < p < \infty$. Then*

$$\lim_{\varepsilon \to 0}^{(L_p)} (\mathbf{D}^\alpha_\varepsilon \mathbf{I}^\alpha \varphi)(x) = \varphi(x).$$

The proof is based on the following lemmas.

Lemma 5.1 *Let $\varphi \in C^\infty_{0,0}$, $\alpha > 0$ and $n \geq 3$. Then*

$$(T_t \mathbf{I}^\alpha \varphi)(x) = \left(J^{\alpha/2} (T_{(\cdot)} \varphi)(x) \right)(t), \qquad t > 0.$$

The proof is a matter of direct verification.

In the following lemma we use the identity approximation kernel

$$k(\eta) = \frac{\Delta^l_1 \left[\eta_+^{\alpha/2} \right]}{\eta \Gamma(\alpha/2 + 1) \varkappa(\alpha/2, l)}\ , \tag{5.2}$$

where

$$(\Delta^l_\tau f)(x) = \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - k\tau),$$

is the finite difference. The kernel $k(\eta)$ is well known in fractional calculus ([17], Subsection 6.2 of Ch. 7). It is known that

$$k(\eta) \in L_1(0, \infty) \quad \text{and} \quad \int_0^\infty k(\eta) \, d\eta = 1. \tag{5.3}$$

Lemma 5.2 *Let $\alpha > 0$, $\varphi \in C_{0,0}^{\infty}$ and $n \geq 3$. Then for any $\varepsilon > 0$ and $t > 0$ the following equality holds:*

$$\frac{1}{\varkappa(\alpha/2,l)}\int_{\varepsilon}^{\infty}\Delta_{\tau}^{l}\left[(T_{(\cdot)}\mathbf{I}^{\alpha}\varphi)(x)\right](t)\frac{d\tau}{\tau^{\alpha/2+1}} =$$
$$= \int_{0}^{\infty} k(\eta)(T_{t+\varepsilon\eta}\varphi)(x)\,d\eta, \qquad 0 < \alpha < 2\ell \ . \tag{5.4}$$

Proof. Relations (5.3) for the kernel (5.2) are well-known (see, for example, [17], Subsection 6.2). The proof of (5.4) is based on the following auxiliary equality

$$\Delta_{\tau}^{l}\left[(T_{(\cdot)}\mathbf{I}^{\alpha}\varphi)(x)\right](t) = \int_{0}^{\infty} g_{\tau}(\xi)(T_{t+\xi}\varphi)(x)\,d\xi, \tag{5.5}$$

where

$$g_{\tau}(\xi) = \frac{1}{\Gamma(\alpha/2)}\Delta_{\tau}^{l}[\xi_{+}^{\alpha/2+1}] \in L_1(0,\infty), \qquad 0 < \alpha < 2l.$$

Indeed, for any $t > 0$ by Lemma 5.1 we have

$$\Delta_{\tau}^{l}\left[(T_{(\cdot)}\mathbf{I}^{\alpha}\varphi)(x)\right](t) = \sum_{k=0}^{l}(-1)^{k}\begin{pmatrix} l \\ k \end{pmatrix}\left(J^{\alpha/2}(T_{t+\xi}\varphi)(x)\right)(t-k\tau) =$$
$$= \sum_{k=0}^{l}(-1)^{k}\begin{pmatrix} l \\ k \end{pmatrix}\frac{1}{\Gamma(\alpha/2)}\int_{0}^{\infty}(T_{\theta}\varphi)(x)(\theta - t + k\tau)_{+}^{\alpha/2-1}d\theta.$$

Setting $\xi = \theta - t$, after simple transformations we obtain (5.5).

To prove (5.4), we take into account (5.5) and for any $\varepsilon > 0$ obtain

$$\frac{1}{\varkappa(\alpha/2,l)}\int_{\varepsilon}^{\infty}\Delta_{\tau}^{l}\left[(T_{(\cdot)}\mathbf{I}^{\alpha}\varphi)(x)\right](t)\frac{d\tau}{\tau^{\alpha/2+1}} =$$
$$= \frac{1}{\varkappa(\alpha/2,l)}\int_{\varepsilon}^{\infty}\left(\int_{0}^{\infty} g_{\tau}(\xi)(T_{t+\xi}\varphi)(x)\,d\xi\right)\frac{d\tau}{\tau^{\alpha/2+1}} =$$
$$= \frac{1}{\varkappa(\alpha/2,l)}\int_{0}^{\infty}(T_{t+\xi}\varphi)(x)\left(\int_{\varepsilon}^{\infty} g_{\tau}(\xi)\frac{d\tau}{\tau^{\alpha/2+1}}\right)d\xi. \tag{5.6}$$

The interchange of order of integration is justified by the Fubini theorem applicable in view of Lemma 4.2. The integral

$$I = \int_{\varepsilon}^{\infty} g_{\tau}(\xi)\frac{d\tau}{\tau^{\alpha/2+1}}.$$

proves to be dilatation of the kernel $k(\xi)$. Indeed, we have

$$\begin{aligned} I &= \frac{1}{\Gamma(\alpha/2)}\int_{\varepsilon}^{\infty}\left(\sum_{k=0}^{l}(-1)^{k}\begin{pmatrix} l \\ k \end{pmatrix}(\xi - k\tau)_{+}^{\alpha/2-1}\right)\frac{d\tau}{\tau^{\alpha/2+1}} = \\ &= \frac{1}{\Gamma(\alpha/2)}\left(-\frac{2}{\alpha}\xi^{\alpha/2-1}\tau^{-\alpha/2}\Big|_{\varepsilon}^{\infty} + \right. \\ &+ \left. \sum_{k=0}^{l}(-1)^{k}\begin{pmatrix} l \\ k \end{pmatrix}k^{\alpha/2}\int_{0}^{(\xi-k\varepsilon)_{+}}\frac{\theta^{\alpha/2-1}\,d\theta}{(\xi-\theta)^{\alpha/2+1}}\right) \end{aligned}$$

and the formula 2.2.5(2) from [10] yields

$$I=\frac{1}{\Gamma(\alpha/2+1)\xi}\sum_{k=0}^{l}(-1)^k\begin{pmatrix} l\\ k\end{pmatrix}\left(\frac{\xi}{\varepsilon}-k\right)_+^{\alpha/2}=\frac{1}{\varepsilon}k\left(\frac{\xi}{\varepsilon}\right).$$

Then from (5.6) we obtain

$$\frac{1}{\varkappa(\alpha/2,l)}\int_{\varepsilon}^{\infty}\Delta_\tau^l\left[(T_{(\cdot)}\mathbf{I}^\alpha\varphi)(x)\right](t)\frac{d\tau}{\tau^{\alpha/2+1}}=\frac{1}{\varkappa(\alpha/2,l)}\int_0^\infty(T_{t+\varepsilon\eta}\varphi)(x)k(\eta)\,d\eta. \tag{5.7}$$

■

The proof of Theorem 5.1. Application of Lemma 5.2 yields

$$(T_t\mathbf{D}_\varepsilon^\alpha\mathbf{I}^\alpha\varphi)(x)=\left(T_t\left[\int_0^\infty k(\eta)(T_{\varepsilon\eta}\varphi)(\cdot)\,d\eta\right]\right)(x),\qquad \varphi\in C_{0,0}^\infty.$$

Passing to the limit as $t\to 0$ in the L_p-norm in the last equality, we have

$$(\mathbf{D}_\varepsilon^\alpha\mathbf{I}^\alpha\varphi)(x)=\int_0^\infty k(\eta)(T_{\varepsilon\eta}\varphi)(x)\,d\eta. \tag{5.8}$$

The equality (5.8) is extended by continuity to the whole space L_p, since the operators on both sides of (5.8) are bounded in L_p. From (5.3) we obtain

$$(\mathbf{D}_\varepsilon^\alpha\mathbf{I}^\alpha\varphi)(x)-\varphi(x)=\int_0^\infty k(\eta)\left[(T_{\varepsilon\eta}\varphi)(x)-\varphi(x)\right]d\eta,$$

with $\varphi\in L_p$, whence

$$\|\mathbf{D}_\varepsilon^\alpha\mathbf{I}^\alpha\varphi-\varphi\|_p\le\int_0^\infty|k(\eta)|\,\|T_{\varepsilon\eta}\varphi-\varphi\|_p\,d\eta\to 0,\quad \varepsilon\to 0$$

by the Lebesgue dominated convergence theorem. (Existence of an integrable dominant follows from (4.9) in view of the assumption $p>n/(n-2)$.) ■

2°. The case $\mathtt{Re}\,\alpha>0$, $\mathtt{Im}\,\alpha\neq 0$. We observe that the approach developed in the previous subsection for real α seems to be problematical in this case. The matter is that we do not know, whether the normalizing constant $\varkappa(\alpha/2,l)$, defined in (5.1), vanishes if $\mathtt{Im}\,\alpha\neq 0$ or not. This is an old problem, equivalent to solvability of some functional equation, which was formulated in [18]. It still remains open. To overcome this difficulty while inverting the potential $f=\mathbf{I}^\alpha\varphi$ in this case, we deal with HSI of the form (1.6) with generalized differences instead of "usual" non-centered ones. See [16], Subsection 6.1 of Ch. 3, for details on these generalized differences. We set

$$\mathbf{D}^\alpha f=\lim_{\varepsilon\to 0}^{(L_p)}\frac{1}{\varkappa(\alpha/2,l)}\int_\varepsilon^\infty(E-T_t)^l f(x)\frac{dt}{t^{\alpha/2+1}},\qquad l>\mathtt{Re}\,\alpha,$$

where

$$(E-T_t)^l f(x)\ =\ \frac{1}{d_l}\begin{vmatrix}(T_{k_0t}f)(x) & 1 & k_0 & \dots & k_0^{l-1}\\ (T_{k_1t}f)(x) & 1 & k_1 & \dots & k_1^{l-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ (T_{k_lt}f)(x) & 1 & k_l & \dots & k_l^{l-1}\end{vmatrix}=\frac{1}{d_l}\sum_{j=0}^{l}c_j(T_{k_jt}f)(x),$$

with

$$d_l = \prod_{l>i>j\geq 0} (k_i - k_j)$$

and

$$\varkappa(\alpha/2, l) = \frac{1}{d_l} \int_0^\infty t^{-\alpha-1} \left(\sum_{j=0}^{l} c_j \exp(-k_j t) \right) dt.$$

The choice $k_j = a^j$, $0 \leq j \leq l$, where $a > 1$ satisfies the condition

$$a \neq \exp(2\pi k/\mathtt{Im}\,\alpha), \qquad k = \pm 1, \pm 2, \dots,$$

made in this subsection, provides the relation $\varkappa(\alpha/2, l) \neq 0$ to be fulfilled (see [16], Lemma 3.38 for details).

Theorem 5.2 *Let* $\mathtt{Re}\,\alpha > 0$, $n \geq 3$ *and* $f = \mathbf{I}^\alpha \varphi$, $\varphi \in L_p$, $n/(n-2) < p < \infty$. *Then* $\mathbf{D}^\alpha f = \varphi$.

The proof is similar to that of Theorem 5.1. It is based on the equality (5.4), where

$$k(\eta) = \frac{1}{\varkappa(\alpha/2, l)\Gamma(1 + \alpha/2)\eta} \sum_{j=0}^{l} c_j (\eta - k_j)_+^\alpha$$

is an identity approximation kernel.

We note that Theorem 5.2 provides an explicit expression for the positive powers $(-|x|^2\Delta)^{\alpha/2} f$, $\mathtt{Re}\,\alpha > 0$ within the framework of L_p-spaces.

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