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On a certain approach to the investigation of equations with involutive operators

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Abstract

An abstract approach to the investigation of Fredholmness of equations $(A_1+A_2Q+\cdots+A_nQ^{n-1})\varphi = f$ in a Banach space is developed, where Q is a generalized involutive operator, that is, $Q^n = I, Q^j \neq I, j = 1, 2, ..., n-1$. Equations with two independent such involutive operators of different orders are also considered. The general results obtained extend the approach given by the authors in previous publications (in Russian). Application is given to two-dimensional singular integral equations with a linear shift of rotational type and to one-dimensional integral equations on R^1 with homogeneous kernels, which include terms both with inversion and complex conjugation.

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In this paper we deal with singular type integral equations which involve the so called involutive operators, that is, the operators Q, which satisfy the condition $Q^2 = I$. More generally, by involutive operator we mean the operator satisfying the condition $Q^n = I$ with some n.

In general, a linear equation with involutive operator has the form

$$(A_1 + QA_2 + \ldots + Q^{n-1}A_n)\varphi = f, \tag{0.1}$$

where $A_j,\ j = 1, ..., n$, have this or that nature. The interest to equations of such a kind has its origin in the theory of singular integral equations with Carleman shift, developed by G.S.Litvinchuk, see his books [13], [12].

In seventieth, the authors developed an approach to the investigation of the Fredholm(=Noether) properties of the equations of the form (1.1) in an abstract Banach space setting, see [5]-[6], [7]. See also [9], where one can find applications of this general approach to various types of integral or discrete equations. Here we present a further modification and simplification of this approach together with its generalization to the case of equations with two independent involutive operators.

We dwell briefly on historical references to previous investigations on the subject. The first consideration of abstract equations with an operator satisfying the condition $Q^2 = I$, was undertaken by Z.Khalilov within the framework of normed rings, see his book [11]. The theory developed there was a direct treating of the theory of singular integral equations with continuous coefficients within the framework of an abstract normed ring. That theory was algebraic in the sense that it was based just on the idea of regularization and did not included any mean to calculate the index. The extension of those results to the case of Banach spaces was given later, see references in [9].

A significant step in the abstract theory was made by Cherskii [1], who constructed the abstract theory of the characteristic singular integral equation. He was first who gave the formula for the index in terms of the so called factorization index of the operator coefficient corresponding to the abstract Riemann boundary problem.

Przeworska-Rolewicz, see the book by Przeworska-Rolewicz [14], investigated equations with an involutive operator or algebraic ($P(Q) = 0, P$ being a polynomial) or almost algebraic operator (P(Q) is compact). There was suggested a simple algebraic approach which allowed to construct regularizers and, in some cases, to obtain solution of the equations.

The investigations carried out in the above mentioned researches were, in fact, based on the model of singular integral operators. By this reason, the theories developed there, did not cover other types of equations with an involutive operator, e.g. singular integral equations with a Carleman shift and even equations with complex conjugate unknowns. In the abstract terms this means that in the equations of the form $(A+QB)\varphi = f$ with $Q^2 = I$ the operators A and B were assumed to be quasicommuting with the involutive operator Q. (In the case of singular integral equations with a Carleman shift this immediately requires invariance of the coefficients of the equations with respect to the shift).

The authors, studying some classes of singular integral equations with a shift and discrete Wiener-Hopf operators with oscillating coefficients, arrived to construction of an abstract theory of the equations $(A + QB)\varphi = f$ in the general non-commutative case. Inspired by the investigations in Karapetiants and Samko [3]-[4], this approach in the first version was presented in Karapetiants and Samko [5],[7], and [8].

The abstract matrix approach has as a prototype the well known idea of the passage from singular integral equations with a Carleman or generalized Carleman shift to a system of such equations without shift, which was widely used in the theory of singular equations, see Litvinchuk [13]. In the form of the corresponding exact matrix identity this idea was given in Gohberg and Krupnik [2] in the case $n = 2$. The generalization of this abstract identity to the case of an arbitrary n, was given in Karapetiants and Samko [7] together with some general approach of investigation of equations with iterations of a generalized involutive operator of order n.

1. Fredholmness of abstract equations with some generalized involutive operators (the matrix approach)

A general method of investigation of Fredholmness of equations with involutive operators presented in this section, is given in two versions: for the case when the equation involves only powers of the same involutive operator Q, and, as a generalization, when the equation may include powers of two independent involutive operators P and Q.

This method has various applications and allows, in particular, to treat the following types of equations: convolution type equations with reflection and complex conjugation, singular integral equations on closed or disclosed curve with a finite group of shifts and discontinuous coefficients, many of such equations being treated in [9].

1.1. The case of one generalized involutive operator

Let X be a Banach space and Q be a generalized involutive operator in X, that is

$$Q^n = I, \quad Q^j \neq I, \quad j = 1, 2, \ldots, n-1, \ n \geq 2.$$

We investigate Fredholm properties of operators of the form

$$K = A_1 + QA_2 + \ldots + Q^{n-1}A_n. \tag{1.1}$$

The operator Q and the "coefficients" $A_j, \ j = 1, 2, \ldots, n$, are assumed to satisfy the following axioms.

$\mathbb{AXIOM}$ 1. *There exists a Fredholm operator $U \in \mathcal{L}(X)$ such that*

$$UQ = \varepsilon_n QU + T, \quad \varepsilon_n = e^{\frac{2\pi i}{n}}, \tag{1.2}$$

where T is compact in X.

$\mathbb{AXIOM}$ 2. *The operators $A_j, \ j = 1, 2, ..., n$ quasicommute with the operator U from the Axiom 1:*

$$A_j U = UA_j + T_j, \quad j = 1, 2, \ldots, n. \tag{1.3}$$

Example 1.1. *Let $X = L_p(R^1), \ 1 < p < \infty$, and $Q\varphi = \varphi(\nu - x)$, where ν is a real number and the coefficients A_j in (1.1) be operators of the form*

$$A_j\varphi = a_j(x)\varphi(x) + c_j(x)(S\varphi)(x) + T_j\varphi, \quad j = 1, 2, \tag{1.4}$$

where $a_j(x), c_j(x) \in C(\dot{R}^1)$ and T_j are compact operators.

We recall that $\dot{R}^1$ is the real line completed by the unique infinite point, so that $a(+\infty) = a(-\infty)$ for $a(x) \in C(\dot{R}^1)$.

The validity of Axiom 1 follows from the equality

$$QS = -SQ\ . \tag{1.5}$$

In view of (1.5), the operator U from Axiom 1 may be chosen as the invertible operator $U = S$. Finally, the validity of Axiom 2 follows from the fact that the commutator $aS - Sa$ is a compact operator in $L_p(R^1),\ 1 < p < \infty$, for any function $a(x) \in C(\dot{R}^1)$.

Example 1.2. *Let Γ be a Liapunov curve and let $X = L_p(\Gamma), 1 < p < \infty$, treated as the space of complex-valued functions over the field of real numbers. We put*

$$Q\varphi = \overline{\varphi(t)}, \quad t \in \Gamma\ . \tag{1.6}$$

and take the operators A_j in the same form (1.4).

Here the operator U from Axiom 1 may be realized as

$$U\varphi = i\varphi(t). \tag{1.7}$$

With the operator (1.1) we relate the following matrix operator acting in $X^n = X \times X \times \ldots X$:

$$\mathbb{K} = \begin{pmatrix} A_1 & QA_2Q^{-1} & Q^2A_3Q^{-2} & \ldots & Q^{n-1}A_nQ^{-n+1} \\ A_2 & QA_3Q^{-1} & Q^2A_4Q^{-2} & \ldots & Q^{n-1}A_1Q^{-n+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ A_n & QA_1Q^{-1} & Q^2A_2Q^{-2} & \ldots & Q^{n-1}A_{n-1}Q^{-n+1} \end{pmatrix}. \tag{1.8}$$

Theorem 1.3. *Fredholmness of the operator $\mathbb{K}$ in X^n is sufficient for that of the operator K in X. Under Axioms 1 and 2 it is also necessary and*

$$Ind_X\ K = \frac{1}{n} Ind_{X^n}\ \mathbb{K}\ . \tag{1.9}$$

Proof. We introduce the operators

$$K^{(s)} = \sum_{j=1}^{n} \varepsilon_n^{s(j-1)} Q^{j-1} A_j$$

and denote

$$V = (\varepsilon_n^{(r-1)(j-1)} I)_{r,j=1}^n, \qquad W = (\delta_{rj} Q^{r-1})_{r,j=1}^n,$$

where δ_{rj} is a Kronecker symbol. The operator W has a diagonal form with invertible operators on the diagonal. The operator V is invertible, since the Vandermonde determinant $det\ (\varepsilon_n^{sk})$ is different from zero. The following equality is valid

$$VW\mathbb{K}WV = n(\delta_{rj} K^{r-1})_{r,j=1}^n, \qquad K^0 = K. \tag{1.10}$$

Since the operators V and W are invertible, the operators $\mathbb{K}$ and $(\delta_{rj} K^{r-1})_{r,j=1}^n$ are simultaneously Fredholm. From Axiom 1 and Axiom 2 we observe that

$$U^s K = K^{(s)} U^s + T_s, \quad s = 1, 2, \ldots, n-1,$$

where T_s are compact operators. Consequently, all the operators $K^{(s)}, \ \ s = 0, 1, \dots, n-1$, are simultaneously Fredholm and their indices coincide.

Sufficiency part. Let the operator $\mathbb{K}$ be Fredholm, then the diagonal operator $(\delta_{rj} K^{r-1})_{r,j=1}^n$ is the same and all the operators $K^{(s)}, \ \ s = 0, 1, \dots, n-1$, are Fredholm. Consequently, the operator K is Fredholm.

Necessity part. Let now the operator K be Fredholm. Then all the operators $K^{(s)}, \ \ s = 0, 1, \dots, n-1$, are Fredholm and $Ind\, K = Ind\, K^{(s)}, \ \ s = 1, 2, \dots, n-1$, so that the diagonal operator $(\delta_{rj} K^{r-1})_{r,j=1}^n$ is also Fredholm and $\mathbb{K}$ is the same. From (1.10) it follows that

$$Ind\ \mathbb{K} = \sum_{s=0}^{n-1} Ind\ K^{(s)} = nInd\ K.$$

□

Remark 1.4. *Let the operator $\mathbb{K}$ be Fredholm in X^n. Then from (1.10) it follows that the operator K and all the operators $K^{(s)}$ are Fredholm in X and*

$$\alpha(\mathbb{K}) = \sum_{s=0}^{n-1} \alpha(K^{(s)}), \quad \beta(\mathbb{K}) = \sum_{s=0}^{n-1} \beta(K^{(s)}).$$

In particular, if the operator $\mathbb{K}$ is invertible (left or right invertible), then the operator K is also invertible (left or right invertible resp.). Let Axioms 1-2 be fulfilled with the additional assumption that the compact operators T and T_j in (1.2)-(1.3) are equal to zero. Then the inverse statement is valid: invertibility of the operator K in X implies that of the operator $\mathbb{K}$ in X^n.

The important particular case $n = 2$ of Theorem 1.3 is given specially in the following theorem. We note that in this case the matrix identity (1.10) turns to be

$$\begin{pmatrix} I & I \\ Q & -Q \end{pmatrix} \begin{pmatrix} A_1 + QA_2 & 0 \\ 0 & A_1 - QA_2 \end{pmatrix} \begin{pmatrix} I & Q \\ I & -Q \end{pmatrix} = 2 \begin{pmatrix} A_1 & QA_2Q \\ A_2 & QA_1Q \end{pmatrix}, \qquad (1.11)$$

where A_1 and A_2 are arbitrary linear operators and $Q^2 = I$, this equality being given in [2].

Theorem 1.5. *Let Axiom 1 and Axiom 2 be fulfilled. The operator $K = A_1 + QA_2$ is Fredholm in X if and only if the operator*

$$\mathbb{K} = \begin{pmatrix} A_1 & QA_2Q \\ A_2 & QA_1Q \end{pmatrix}$$

is Fredholm in X^2. In this case $Ind_X\ K = \frac{1}{2} Ind_{X^2}\ \mathbb{K}$.

Theorem 1.3 may be easily transformed to the case when the operators A_j in (1.1) are matrix operators. For the formulation of the result of such a kind we introduce the matrix operators $\widehat{Q} = (\delta_{rk} Q)_{r,k=1}^m, \quad \widehat{A}_j = \left(A_{rk}^{(j)} \right)_{r,k=1}^m$, acting in $X^m = X \times X \times \dots X$. We consider the operator

$$\widehat{K} = \sum_{j=1}^n \widehat{Q}^{j-1} \widehat{A}_j.$$

As above, we introduce the matrix operator $\widehat{\mathbb{K}}$ related to the operator $\widehat{K}$:

$$\widehat{\mathbb{K}} = (\widehat{Q}^{r-1}\widehat{A}_{r+j-1}\widehat{Q}^{-(j-1)})_{r,j=1}^{n}, \qquad \widehat{A}_{n+s} = \widehat{A}_s.$$

Theorem 1.6. *Let the operators $A_{rk}^{(j)}$ satisfy Axioms 1 and 2 . The operator $\widehat{K}$ is Fredholm in X^m if and only if the operator $\widehat{\mathbb{K}}$ is Fredholm in X^{mn}. In this case*

$$Ind_{X^m}\ \widehat{K} = \frac{1}{n} Ind_{X^{mn}}\ \widehat{\mathbb{K}}\ . \tag{1.12}$$

Proof. It is evident that $\widehat{Q}$ is an involutive operator. Applying Theorem 1.3 to the operator $\widehat{K}$ we should verify that Axiom 1 and Axiom 2 are fulfilled for the operator $\widehat{Q}$ and the "coefficients" $\widehat{A}_j$. If we put $\widehat{U} = (\delta_{rj}U)_{r,j=1}^{m}$, then we see that Axiom 1 yields $\widehat{Q}\widehat{U} = \varepsilon_n \widehat{Q}\widehat{U} + T$, where T is compact in X^m. Consequently, Axiom 1 is satisfied. The validity of Axiom 2 is evident. □

1.2. The case of two generalized involutive operators

Let X be a Banach space and let P and Q be two generalized involutive operators of orders n and m respectively:

$$\begin{gathered} Q^n = P^m = I, \\ P^j \neq I, \quad j = 1, 2, \ldots, m-1; \quad Q^k \neq I,\ k = 1, 2, \ldots, n-1. \end{gathered} \tag{1.13}$$

We study the Fredholm properties of the operator K of the form

$$K = \sum_{j=1}^{m}\sum_{k=1}^{n} P^{j-1}Q^{k-1}A_{jk}. \tag{1.14}$$

a) The general case. The involutive operators P and Q and the "coefficients" A_{jk} are supposed to satisfy the following axioms.

$\mathbb{AXIOM}$ 1. *There exist Fredholm operators $U_P \in \mathcal{L}(X)$ and $U_Q \in \mathcal{L}(X)$ such that*

$$U_P P = \varepsilon_m P U_P + T_1, \qquad \varepsilon_m = e^{\frac{2\pi i}{m}}, \tag{1.15}$$

$$U_Q P = \varepsilon_n Q U_P + T_2, \qquad \varepsilon_n = e^{\frac{2\pi i}{n}}, \tag{1.16}$$

$$U_P Q = Q U_P + T_3, \tag{1.17}$$

where $T_j,\ j = 1, 2, 3$ are compact in X.

$\mathbb{AXIOM}$ 2. *The following quasicommutation relations for the operators U_P and U_Q hold*

$$U_P A_{jk} = A_{jk} U_P + T_4, \tag{1.18}$$

$$U_Q A'_{jk} = A'_{jk} U_Q + T_5, \tag{1.19}$$

where $A'_{jk} = P^l A_{jk} P^{-l},\ j = 1, 2, \ldots, m;\quad k = 1, 2, \ldots, n;\quad l = 0, 1, \ldots, m-1.$

In the case when P and Q commute, we could apply our approach of Subsection 1.2 "with respect to the operator Q". However, we shall treat the operator K under the assumption more general than the condition of commutativity of P and Q. Namely, we suppose that the following axiom is satisfied.

AXIOM 3. *There exist a real number γ and an integer ν such that*

$$PQP^{-1} = \gamma Q^{\nu}. \tag{1.20}$$

As we shall see below, this axiom is realized in applications with some ν and γ just in this form. We note some corollaries of this axiom:

1). The number γ may be only a root of 1: $\gamma^n = 1$.

2). The following equality holds

$$P^jQ^kP^{-j} = \gamma^{k(1+\nu+\ldots+\nu^{j-1})}Q^{k\nu^j}. \tag{1.21}$$

3). Independently of n, the $(\nu^m - 1)$-th power of Q is reduced to the multiplication by some power of ε_n:

$$Q^{\nu^m-1} = \gamma^{-\frac{\nu^m-1}{\nu-1}}I; \tag{1.22}$$

if $(\nu, n) = p$, then $Q^{\frac{n}{p}} = \gamma^{\frac{n}{p}}I$.

4). If the order m is even and $\nu = -1$, then the operators P and Q is commute.

5). The group, generated by the operators P and Q and their powers is finite.

Proof. 1). The equality $\gamma^n = 1$ is obtained by raising (1.20) to the power (1.20), taking the equality

$$(PQP^{-1})^j = PQ^jP^{-1} \tag{1.23}$$

into account.

2). In view of (1.23), it is sufficient to prove the formula (1.21) for $k = 1$ only. We use the induction method. If $j = 1$, the formula (1.21) is evident. Assuming that it is true for some j, we have $P^{j+1}Q^kP^{-(j+1)} = P\gamma^{1+\nu+\ldots+\nu^{j-1}}Q^{\nu^j}P^{-1} = \gamma^{1+\nu+\ldots+\nu^{j-1}+\nu^j)}Q^{\nu^{j+1}}$, and then it is also true for the number $j + 1$.

3). The equality (1.22) is obtained from (1.21), if we put $k = 1$ and $j = m$. Let $(\nu, n) = p$. By raising the equality (1.20) to the power $\frac{n}{p}$, we have $PQ^{\frac{n}{p}}P^{-1} = \gamma^{\frac{n}{p}}Q^{\frac{\nu}{p}+n}$. Since $\frac{\nu}{p}$ is an integer, we obtain $PQ^{\frac{n}{p}}P^{-1} = \gamma^{\frac{n}{p}}I$, which yields $PQ^{\frac{n}{p}} = \gamma^{\frac{n}{p}}P$.

4). If $\nu = -1$ and the order m is even, from (1.22) we have $Q^{-2} = \gamma^{-1}I$ and then $\gamma Q^{-1} = Q$. The last equality yields $PQP^{-1} = Q$ or $PQ = QP$.

5). The proof is direct.

□

We return to the investigation of the operator (1.14) under the assumption that Axioms 1-3 are fulfilled.

We rewrite the operator (1.14) in the form

$$K = \sum_{j=1}^{m} P^{j-1}B_j, \tag{1.24}$$

where

$$B_j = \sum_{k=1}^{n} Q^{k-1} A_{jk}.$$

We apply Theorem 1.3 to this operator with respect to the involutive operator P with the "coefficients" B_j. It is easy to see that Axioms 1-2 from Subsection 1.2 are fulfilled in view of (1.15),(1.17) and (1.18). By Theorem 1.3, the operator (1.24) is Fredholm simultaneously with the matrix operator

$$\mathbb{K} = (P^{j-1} B_{r+j-1} P^{-(j-1)})_{r,j=1}^{m} = (N_{rj})_{r,j=1}^{m}, \tag{1.25}$$

where

$$N_{rj} = \sum_{j=1}^{m} P^{j-1} Q^{k-1} A_{r+j-1,k} P^{-(j-1)}. \tag{1.26}$$

In the notation B_{r+j-1} it is assumed that $B_{j+m} \equiv B_j$and similarly for $A_{r+j-1,k}$.

Up to now we did not use Axiom 3 and obtained the following preliminary result.

Theorem 1.7. *Let the operators A_{jk} satisfy Axioms 1 and 2 . The operator K is Fredholm in X if and only if the operator $\mathbb{K}$ is Fredholm in X^m and*

$$Ind_X\ K = \frac{1}{m} Ind_{X^m}\ \mathbb{K}\ . \tag{1.27}$$

To obtain the final result, we suppose now that Axiom 3 is also fulfilled. Using the formula (1.21), we rewrite the matrix operator (1.25) in the form

$$\mathbb{K} = \sum_{k=1}^{n} \left(Q^{(k-1)\nu^{j-1}} P^{j-1} \gamma^{(k-1)\frac{1-\nu^{j-1}}{1-\nu}} A_{r+j-1,k} P^{-(j-1)} \right)_{r,j=1}^{n} = \sum_{k=1}^{n} \widehat{N}_k.$$

It may be represented as

$$\mathbb{K} = \sum_{k=1}^{n} \widehat{Q}^{k-1} \widehat{Z}_k, \qquad \widehat{Q} = (\delta_{rj} Q)_{r,j=1}^{n},$$

where the matrix $\widehat{Z}_k$ consists of elements $P^{j-1} A P^{-(j-1)}$, where each operator A is a linear combination of the operators A_{jk} . Now we apply Theorem 1.6 with respect to the involutive operator $\widetilde{Q}$ with the coefficients $\widehat{Z}_k$. It is evident, that Axioms 1 and 2 from the Subsection 1.2 with respect to $\widetilde{Q}$ and $\widehat{Z}_k$ are fulfilled and we have the following final result.

Theorem 1.8. *Let the operators A_{jk} satisfy Axioms 1-3 . The operator K is Fredholm in X if and only if the operator*

$$\widehat{\mathbb{K}} = \left(\widehat{Q}^{(\mu-1)} \widehat{Z}_{\lambda+\mu-1} \widehat{Q}^{1-\mu)} \right)_{\lambda,\mu=1}^{n} = (M_{\lambda,\mu})_{\lambda,\mu=1}^{n} \tag{1.28}$$

is Fredholm in X^{mn} and

$$Ind_X\ K = \frac{1}{mn} Ind_{X^{mn}}\ \widehat{\mathbb{K}}\ . \tag{1.29}$$

We can construct the operators $\widehat{Z}_k$ effectively in two cases: a) $\nu = 1$ or b) $\nu = -1$ in Axiom 3.

The case a). In this case Axiom 3 takes the form $PQ = \gamma QP$ and we see, that

$$\widehat{Z}_k = (\gamma^{(k-1)(j-1)} P^{j-1} A_{r+j-1,k} P^{-(j-1)})_{r,j=1}^m.$$

Consequently,

$$M_{\lambda\mu} = (\gamma^{(j-1)(\lambda+\mu-2)} Q^{\mu-1} P^{j-1} A_{r+j-1,\lambda+\mu-1} P^{-(j-1)} Q^{-(\mu-1)})_{r,j=1}^n \tag{1.30}$$

in this case .

The case b). In this case Axiom 3 is $PQ = \gamma Q^{-1}P$ and the matrix $\widehat{Z}_k$ contains odd columns of the matrix $\widehat{N}_k$ and even columns of the matrix $\widehat{N}_{n-k+2}$. Hence

$$\widehat{Z}_k = \left(\frac{1+(-1)^{j-1}}{2}\gamma^{(k-1)\frac{1+(-1)^j}{2}} P^{j-1} A_{r+j-1,k} P^{-j+1}\right)_{r,j=1}^m +$$

$$+ \left(\frac{1+(-1)^{j}}{2}\gamma^{(n-k-1)\frac{1+(-1)^j}{2}} P^{j-1} A_{r+j-1,n-k+2} P^{-j+1}\right)_{r,j=1}^m,$$

so that

$$\begin{aligned} M_{\lambda\mu} = &\left(\tfrac{1+(-1)^{j-1}}{2}\gamma^{(\lambda+\mu-2)\frac{1+(-1)^j}{2}} Q^{\mu-1} P^{j-1} A_{r+j-1,\lambda+\mu-1} P^{-j+1} Q^{-\mu+1}\right)_{r,j=1}^m + \\ &\left(\tfrac{1+(-1)^{j}}{2}\gamma^{(n-\lambda-\mu-2)\frac{1+(-1)^j}{2}} Q^{\mu-1} P^{j-1} A_{r+j-1,n-\lambda-\mu+3} P^{-j+1} Q^{-\mu+1}\right)_{r,j=1}^m \end{aligned} \tag{1.31}$$

in this case.

b) The case $m = n = 2$. We single out the case when both of involutions have the order 2 and suppose that $PQ = QP$ keeping applications in mind.

In this case the operator (1.14) has the form

$$K = (A_{11} + QA_{12}) + P(A_{21} + QA_{22}). \tag{1.32}$$

We suppose that Axioms 1 and 2 are satisfied. To construct the matrix operator without shift, we observe that at the first step we obtain the following matrix operator

$$\begin{pmatrix} A_{11} + QA_{12} & P(A_{21} + QA_{22})P \\ A_{21} + QA_{22} & P(A_{11} + QA_{12})P \end{pmatrix} \tag{1.33}$$

which includes the involutive matrix operator $\widehat{Q}$ of order 2:

$$\begin{pmatrix} A_{11} & PA_{21}P \\ A_{21} & PA_{11}P \end{pmatrix} + \widehat{Q} \begin{pmatrix} A_{12} & PA_{22}P \\ A_{22} & PA_{12}P \end{pmatrix}, \quad \widehat{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}. \tag{1.34}$$

At the next step, we arrive at the matrix operator

$$\widehat{\mathbb{K}} = \begin{pmatrix} A_{11} & PA_{21}P & QA_{12}Q & QPA_{22}PQ \\ A_{21} & PA_{11}P & QA_{22}Q & QPA_{12}PQ \\ A_{12} & PA_{22}P & QA_{11}Q & QPA_{21}PQ \\ A_{22} & PA_{12}P & QA_{21}Q & QPA_{11}PQ \end{pmatrix} . \tag{1.35}$$

As a corollary of Theorem 1.8 we obtain the following result.

Theorem 1.9. *Let the operators A_{jk} satisfy Axioms 1-2 and PQ=QP. The operator K of the form (1.32) is Fredholm in X if and only if the operator $\widehat{\mathbb{K}}$ of the form (1.35) is Fredholm in X^4 and in this case*

$$Ind_X\ K = \frac{1}{4} Ind_{X^4}\ \widehat{\mathbb{K}} . \tag{1.36}$$

2. Application to integral equations

2.1. Calderon-Zygmund operators with linear Carleman shift

We shall consider two-dimensional singular integral equations with Calderon-Zygmund singular operators of the type

$$(\mathcal{S}\varphi)(x) = \int_{R^2} \frac{\Omega(y')}{|y|^2} \varphi(x-y)dy, \quad x \in R^2, \ \ y' = \frac{y}{|y|} , \tag{2.1}$$

and linear Carleman type shift. By $\alpha(x) = \mathfrak{A}x + \beta$ we denote a linear transformation on R^2, satisfying the generalized Carleman condition of order $n \geq 2$ (that is, $\alpha_n(x) \equiv x,\ \alpha_n(x) = \alpha[\alpha_{n-1}(x)]$), generated by an orthogonal matrix $\mathfrak{A}$. The integral operator under the consideration will be

$$(K\varphi)(x) = \sum_{j=0}^{n-1} \left\{ a_j \varphi[\alpha_j(x)] + b_j (\mathcal{S}_{\Omega_j}\varphi)[\alpha_j(x)] \right\} = f(x), \quad x \in R^2, \tag{2.2}$$

where a_j, b_j are constants and $\alpha_0(x) = x$. It is assumed that $\Omega_j\left(\frac{x}{|x|}\right),\ j = 0, 1, \ldots, n-1$, satisfy the standard conditions which provide the boundedness of the corresponding Calderon-Zygmund operators in the space $L_p(R^2), 1 < p < \infty$.

The function

$$\sigma(\xi') = \int_{|t|=1} \ln \frac{1}{-\xi' \cdot t}\ \Omega(t)dt, \quad |\xi'| = 1,$$

is known as the *symbol* of the singular operator $\mathcal{S}_\Omega$.

Lemma 2.1. *Let A be any orthogonal linear transformation in R^2 and $Q\varphi = \varphi(Ax + \beta),\ \beta \in R^n$. Then*

$$Q\mathcal{S}_\Omega Q^{-1} = \mathcal{S}_{\Omega^*} \quad \text{with} \quad \Omega^*(x) = \Omega(Ax). \tag{2.3}$$

The operator (2.2) has the form

$$K = \sum_{j=1}^{n} Q^{j-1} A_j \quad \text{with} \quad A_j = a_{j-1} + b_{j-1}\mathcal{S}_{\Omega_{j-1}}, \quad j = 1, 2, ..., n. \tag{2.4}$$

This enables us to apply the general Theorem 1.3.

We denote

$$\Sigma(\xi) = \mathcal{A} + \mathcal{H}(\xi), \quad \text{where} \quad \mathcal{A} = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_1 & a_2 & \dots & a_0 \\ \dots & \dots & \dots & \dots \\ a_{n-1} & a_0 & \dots & a_{n-2} \end{pmatrix}$$

and

$$\mathcal{H}(\xi) = \begin{pmatrix} b_0\sigma_0(\xi) & b_1\sigma_1(\mathfrak{A}\xi) & \dots & b_{n-1}\sigma_{n-1}(\mathfrak{A}^{n-1}\xi) \\ b_1\sigma_1(\xi) & b_2\sigma_2(\mathfrak{A}\xi) & \dots & b_0\sigma_0(\mathfrak{A}^{n-1}\xi) \\ \dots & \dots & \dots & \dots \\ b_{n-1}\sigma_{n-1}(\xi) & b_0\sigma_0(\mathfrak{A}\xi) & \dots & b_{n-2}\sigma_{n-2}(\mathfrak{A}^{n-1}\xi) \end{pmatrix} \tag{2.5}$$

where $\sigma_j(\xi)$ are the symbols of Calderon-Zygmund operators with the characteristics $\Omega_j(x')$.

Theorem 2.2. *The operator (2.2) is invertible in the space $L_p(R^2), 1 < p < \infty$, if and only if* $\inf_{|\xi|=1} |\det \Sigma(\xi)| \neq 0$.

Proof. Any linear Carleman-type transformation may be reduced to the so called canonical case which is either the "rational" rotation or reflection with respect to one of the variables. Therefore, one may take from the very beginning

$$\mathfrak{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad \text{with} \quad \theta = \frac{2\pi k}{n} \quad \textit{or} \quad \mathfrak{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider the first possibility as more interesting, the second one being easier. To apply Theorem 1.3, we have to construct the operator U satisfying Axioms 1-2 of that theorem. We look for the operator U in the form of a singular Calderon-Zygmund operator $U = \mathcal{S}_\Omega$ with some characteristic $\Omega(x')$. Axiom 1 is then reduced to the following relation for the characteristic $\Omega(x')$:

$$\Omega(y) = \varepsilon_n \Omega(\mathfrak{A}y). \tag{2.6}$$

We take $k = 1$ for simplicity and choose $\Omega(y) = \frac{y_1 + iy_2}{|y|}$ (the idea of this choice in a different situation was formulated by A.I.Koshelenko, Master Theses, Rostov State University, 1981). Then the symbol of the operator U is equal to

$$\sigma(\xi) = \frac{i\pi^2}{2}(\xi_1 + i\xi_2) \ , \qquad \text{so that} \qquad \sigma(\xi) \neq 0 \quad \text{for} \quad |\xi| = 1 \ .$$

Therefore, the operator U is invertible. It commutes with the operators A_k as a convolution operator. Therefore, Axioms 1 and 2 of Theorem 1.3 are satisfied. The entries $Q^{j-1}A_{r+j-1}Q^{1-j}$ in the matrix operator (1.8) from Theorem 1.3 are equal to

$$Q^{j-1}A_{r+j-1}Q^{1-j}\varphi = a_{r+j-2}\varphi(x) + b_{r+j-2}\mathcal{S}_{r,j} \ ,$$

where $\mathcal{S}_{r,j}$ is the Calderon-Zygmund singular operator with the characteristic $\Omega_{r+j-2}(\mathfrak{A}^{j-1}x)$, computed in accordance with Lemma 2.1. Calculating the symbol of this matrix operator and applying Theorem 1.3, we arrive at the conclusion of our theorem. □

2.2. Singular integral operators with homogeneous kernels

This section is preliminary. We recall here some results on Fredholmness of $m \times m$-systems of integral equations on R^1 with homogeneous kernels of degree -1. The scalar equation ($m = 1$) has the form

$$N\varphi \equiv a(x)\varphi(x) + b(x)\int_{-\infty}^{\infty} k(x,y)\varphi(y)\,dy = f(x), \tag{2.7}$$

where

$$k(\lambda x, \lambda y) = \lambda^{-1}k(x,y), \;\; \lambda > 0.$$

We consider the operator (2.7) in the space $L_p(R^1)$, $1 < p < \infty$, and assume that $a(x),\, b(x) \in C(\dot{R}^1)$ and

$$\int_{-\infty}^{\infty} |y|^{-\frac{1}{p}}|k(\pm 1, y)|dy < \infty, \tag{2.8}$$

the latter condition guaranteeing the boundedness of the integral operator in (2.7) in the space $L_p(R^1)$. This condition obviously excludes singular homogeneous kernels of the type $k(x,y) = \frac{1}{x-y}$. We recall that equations with homogeneous kernels are easily reduced to convolution type equations by means of direct exponential change of variables. We give the corresponding result for systems of equations with homogeneous kernels, see Theorem 2.3. This theorem may be considered as well known, see for example, [10] in the case $m = 1$.

In what follows the notation

$$\mathcal{K}_{\pm\pm}(\xi) = \int_0^{\infty} k(\pm 1, \pm y)y^{\xi-1}dy \tag{2.9}$$

will stand for the Mellin transforms of the kernels $k(\pm 1, \pm y)$.

The corresponding system of integral equations with homogeneous kernels has the form

$$N\varphi \equiv A(x)\varphi(x) + B(x)\int_{-\infty}^{\infty} K(x,y)\varphi(y)\,dy = F(x), \tag{2.10}$$

where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m)$ is a vector-function, $A(x), B(x)$ and $K(x,y)$ are $(m \times m)$-matrices. We assume that the entries of the matrices $A(x)$ and $B(x)$ are continuous on $\dot{R}^1$, and the kernel $K(x,y)$ has entries satisfying the conditions (2.8).

It may be shown that Fredholmness of the matrix operator AI is necessary for that of the operator N. By this reason, we assume that $\det A(x) \neq 0$, $x \in \dot{R}^1$. We denote

$$\sigma_0(\xi) = \begin{pmatrix} \sigma_0^{11}(\xi) & \sigma_0^{12}(\xi) \\ \\ \sigma_0^{21}(\xi) & \sigma_0^{22}(\xi) \end{pmatrix}, \quad \sigma_\infty(\xi) = \begin{pmatrix} \sigma_\infty^{11}(\xi) & \sigma_\infty^{12}(\xi) \\ \\ \sigma_\infty^{21}(\xi) & \sigma_\infty^{22}(\xi) \end{pmatrix}, \tag{2.11}$$

where the $(m \times m)$-blocs $\sigma_0^{kj}(\xi)$ and $\sigma_\infty^{kj}(\xi)$ have the form:

$$\sigma_0^{11}(\xi) = I + A^{-1}(0)B(0)K_{++}(\xi), \quad \sigma_0^{12}(\xi) = A^{-1}(0)B(0)K_{+-}(\xi)\;,$$

$$\sigma_0^{21}(\xi) = A^{-1}(0)B(0)K_{-+}(\xi)\ , \quad \sigma_0^{22}(\xi) = I + A^{-1}(0)B(0)K_{--}(\xi) \tag{2.12}$$

and similarly for $\sigma_\infty^{kj}(\xi),\ k,j=1,2$ with $A^{-1}(0)B(0)$ replaced by $A^{-1}(\infty)B(\infty)$.

Here $K_{\pm\pm}(\xi)$ are matrices with elements of the form (2.9) calculated for the kernels $k_{jr}(x,y)$.

Theorem 2.3. *Let the entries of the matrices $A(x)$ and $B(x)$ be in $C(\dot{R}^1)$ and the entries of the matrix $K(x,y)$ satisfy the conditions (2.8). The operator of the form (2.10) is Fredholm in the space $L_p(R^1),\ 1<p<\infty,$ if and only if* $\det A(x)\neq 0$ *and*

$$\det\sigma_0\left(ix+1-\frac{1}{p}\right)\neq 0\ , \quad \det\sigma_\infty\left(ix+1-\frac{1}{p}\right)\neq 0$$

for all $x\in\dot{R}^1$. Under these conditions

$$\mathrm{Ind}_{L_p}N \ = \ \mathrm{ind}\frac{\det\sigma_\infty\left(ix+1-\frac{1}{p}\right)}{\det\sigma_0\left(ix+1-\frac{1}{p}\right)}.$$

2.3. Integral equations with homogeneous kernels involving terms with inversion and complex conjugation.

Let us consider the following integral equation

$$K\varphi :\equiv a(x)\varphi(x) + b(x)\overline{\varphi(x)} + c(x)\varphi\left(\tfrac{1}{x}\right) + d(x)\overline{\varphi\left(\tfrac{1}{x}\right)} + \alpha(x)\int\limits_{-\infty}^{\infty} k(x,y)\varphi(y)dy$$

$$+\beta(x)\int\limits_{-\infty}^{\infty}\ell(x,y)\overline{\varphi(y)}dy + \gamma(x)\int\limits_{-\infty}^{\infty}p(x,y)\varphi\left(\tfrac{1}{y}\right)dy + \delta(x)\int\limits_{-\infty}^{\infty}q(x,y)\overline{\varphi\left(\tfrac{1}{y}\right)}dy = f(x), \tag{2.13}$$

where $k(x,y), l(x,y), p(x,y)$ and $q(x,y)$ are homogeneous kernels of degree -1.

Here we have two involutions $\overline{\varphi(x)}$ and $\varphi\left(\frac{1}{x}\right)$. To have a bounded involutive operator in the latter case, we introduce it in the form $(Q\varphi)(x) = |x|^{-\frac{2}{p}}\varphi\left(\frac{1}{x}\right)$. In this case $\|Q\varphi\|_p = \|\varphi\|_p$. To be able to apply the general Theorem 1.9, we rewrite the equation (2.13) introducing the following notation:

$$a_{11}(x) = a(x) \qquad a_{12}(x) = |x|^{-\frac{2}{p}}c\left(\tfrac{1}{x}\right) \qquad a_{21}(x) = \overline{b(x)} \qquad a_{22}(x) = |x|^{-\frac{2}{p}}\overline{d\left(\tfrac{1}{x}\right)}$$

$$b_{11}(x) = \alpha(x) \qquad b_{12}(x) = |x|^{-\frac{2}{p}}\gamma\left(\tfrac{1}{x}\right) \qquad b_{21}(x) = \overline{\beta(x)} \qquad b_{22}(x) = |x|^{-\frac{2}{p}}\overline{\delta\left(\tfrac{1}{x}\right)}$$

$$k_{11}(x,y) = k(x,y) \quad k_{12}(x,y) = \tfrac{1}{y^2}p\left(\tfrac{1}{x},\tfrac{1}{y}\right) \quad k_{21}(x,y) = \overline{\ell(x,y)} \quad k_{22}(x,y) = \tfrac{1}{y^2}\overline{q\left(\tfrac{1}{x},\tfrac{1}{y}\right)}$$

and

$$Q\varphi(x) = \frac{1}{|x|^{\frac{2}{p}}}\varphi\left(\frac{1}{x}\right), \quad P\varphi(x) = \overline{\varphi(x)}. \tag{2.14}$$

Evidently, $Q^2 = P^2 = I$ and $QP = PQ$. Under these notations, the equation (2.13) may be rewritten as

$$K = (N_{11} + QN_{12}) + P(N_{21} + QN_{22}), \tag{2.15}$$

where N_{jr} are operators of the form (2.7):

$$N_{jr} = a_{jr}(x)I + c_{jr}(x)K_{jr}, \quad j, r = 1, 2.$$

We assume that

$$a_{jr}(x), \quad b_{jr}(x) \in C(\dot{R}^1) \tag{2.16}$$

$$\int\limits_{-\infty}^{\infty} |y|^{-\frac{1}{p}} |k_{jr}(\pm 1, y)| dy < \infty, \quad j, r, = 1, 2. \tag{2.17}$$

We consider the operator K in the space $L_p(R^1)$. Axioms 1 and 2 from Subsection 1.2 are satisfied under the choice

$$U_P \varphi(x) = i\varphi(x), \quad U_Q \varphi(x) = sign(\ln|x|)\varphi(x).$$

The relations

$$U_P P + PU_P = 0, \quad \text{and} \quad U_Q Q + QU_Q = 0$$

are obvious, so that Axiom 1 is satisfied. As regards Axiom 2, we evidently have $U_P A_{jr} = A_{jr} U_P$ and $U_Q A_{jr} = A_{jr} U_Q + T_{jr}$ where T_{jr} are compact operators. Here we used the fact that the operator

$$sign(\ln|x|)K_{jr} - K_{jr} sign(\ln|y|)$$

is compact under our conditions, see [10]. To formulate the final result we need to calculate the entries of the matrix operator (1.35). We have

$$PN_{jr}P = \overline{a_{jr}(x)}I + \overline{c_{jr}(x)}\overline{K_{jr}}, \quad \overline{K_{jr}}\varphi(x) = \int_{-\infty}^{\infty} \overline{k_{jr}(x, y)}\varphi(y)dy, \tag{2.18}$$

$$QN_{jr}Q = a^*_{jr}(x)I + c^*_{jr}(x)K^*_{jr}, \quad K^*_{jr}\varphi(x) = \int_{-\infty}^{\infty} k^*_{jr}(x, y)\varphi(y)dy, \tag{2.19}$$

where

$$k^*_{jr}(x, y) = \frac{1}{y^2} \left|\frac{y}{x}\right|^{\frac{2}{p}} k\left(\frac{1}{x}, \frac{1}{y}\right)$$

and

$$QPN_{jr}PQ = \overline{a^*_{jr}(x)}I + \overline{c^*_{jr}(x)}\overline{K^*_{jr}}, \quad \overline{K^*_{jr}}\varphi(x) = \int_{-\infty}^{\infty} \overline{k^*_{jr}(x, y)}\varphi(y)dy \tag{2.20}$$

for all $j, r = 1, 2$.

It is easy to check that the kernels $k^*_{jr}(x, y)$ satisfy the same integrability condition (2.8) as $k_{jr}(x, y)$.

Theorem 1.9 leads us to the matrix operator of the form (2.10) which has to be considered in the space $L_p^4(R^1)$, where

$$A(x) = \begin{pmatrix} a_{11}(x) & \overline{a_{21}(x)} & a_{12}^*(x) & \overline{a_{22}^*(x)} \\ a_{21}(x) & \overline{a_{11}(x)} & a_{22}^*(x) & \overline{a_{12}^*(x)} \\ a_{12}(x) & \overline{a_{22}(x)} & a_{11}^*(x) & \overline{a_{21}^*(x)} \\ a_{22}(x) & \overline{a_{12}(x)} & a_{21}^*(x) & \overline{a_{11}^*(x)} \end{pmatrix}, \quad B(x) = \begin{pmatrix} b_{11}(x) & \overline{b_{21}(x)} & b_{12}^*(x) & \overline{b_{22}^*(x)} \\ b_{21}(x) & \overline{b_{11}(x)} & b_{22}^*(x) & \overline{b_{12}^*(x)} \\ b_{12}(x) & \overline{b_{22}(x)} & b_{11}^*(x) & \overline{b_{21}^*(x)} \\ b_{22}(x) & \overline{b_{12}(x)} & b_{21}^*(x) & \overline{b_{11}^*(x)} \end{pmatrix}$$

and

$$K(x,y) = \begin{pmatrix} k_{11}(x,y) & \overline{k_{21}(x,y)} & k_{12}^*(x,y) & \overline{k_{22}^*(x,y)} \\ k_{21}(x,y) & \overline{k_{11}(x,y)} & k_{22}^*(x,y) & \overline{k_{12}^*(x,y)} \\ k_{12}(x,y) & \overline{k_{22}(x,y)} & k_{11}^*(x,y) & \overline{k_{21}^*(x,y)} \\ k_{22}(x,y) & \overline{k_{12}(x,y)} & k_{21}^*(x,y) & \overline{k_{11}^*(x,y)} \end{pmatrix}. \tag{2.21}$$

It remains to apply Theorem 2.3 and we arrive at the following result.

Theorem 2.4. *Under the assumptions (2.16) and (2.17), the operator (2.13) is Fredholm in the space* $L_p(R^1)$, $1 < p < \infty$, *if and only if* $\det A(x) \neq 0$ *and*

$$\det \sigma_0 \left(ix + 1 - \frac{1}{p} \right) \neq 0 \ , \quad \det \sigma_\infty \left(ix + 1 - \frac{1}{p} \right) \neq 0$$

for all $x \in \dot{R}^1$, *where the* 8×8*-matrices* $\sigma_0(\xi)$ *and* $\sigma_\infty(\xi)$ *are given by (2.11) and (2.12) with* $K(x,y)$ *defined in (2.21). Under these conditions*

$$\mathrm{Ind}_{L_p} K \ = \ \frac{1}{2} \mathrm{ind} \frac{\det \sigma_\infty \left(ix + 1 - \frac{1}{p} \right)}{\det \sigma_0 \left(ix + 1 - \frac{1}{p} \right)}. \tag{2.22}$$

Remark. Theorem 1.9 gives the fraction $\frac{1}{4}$ for the formula for the index. But we have $\frac{1}{2}$ in (2.22), because we take into account only real coefficients while considering linear combinations of complex-valued functions.

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