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Convolution and potential type operators in $L^{p(x)}(R^n)$.

Introduction

In this paper we give a further development of the results of the paper [1] and apply it to convolution operators

$$Kf = k * f = \int_{R^n} k(x-y)f(y)dy \tag{1}$$

in the spaces $L^{p(x)}$. We consider the question of extendability of the Young theorem : $\|Kf\|_r \le \|k\|_q \|f\|_p$, $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, well known for constant p and q , to the case when they may be variable. We also treat potential type operators with the kernel $| \, x-y \, |^{\alpha(x)-n}$.

In Section 1 we develop some estimates for $L^{p(x)}$-norms of power functions of distance truncated to exterior of a ball of radius $r > 0$, as $r \to 0$ or $r \to \infty$. Section 2 deals with convolution operators in the spaces $L^{p(x)}$ and Section 3 is devoted to potential type operators.

I. Estimates for $L^{p(x)}$-norms of power functions of distance truncated to exterior of a ball

We use notations from [1], in particular:

Ω is a measurable set in R^n, $|\Omega|$ is its Lebesgue measure, $\dot{R}^n$ is the compactification of R^n by the unique infinite point; $S_{n-1} = \{x \in R^n : |x| = 1\}$; $B(x_0, r)$ is the ball centered at x_0 and of radius r ; $B_n = B(0,1)$; $p(x_0, r) =$

$\inf_{|y-x_0|\le r} p(y), P(x_0,r) = \sup_{|y-x_0|\le r} p(y), \bar{p}(x_0,r) = \inf_{|y-x_0|\ge r} p(y), \bar{P}(x_0,r) = \sup_{|y-x_0|\ge r} p(y); \chi_\Omega(x) \equiv 1$ for $x \in \Omega$ and $\equiv 0$ for $x \notin \Omega$; $f_\beta(x) = f_\beta(x_0,r;x) = | x-x_0 |^{\beta(x_0)} \chi_{B(x_0,r)}(x), g_\beta(x) = g_\beta(x_0,r;x) =| x-x_0 |^{\beta(x_0)} [1-\chi_{B(x_0,r)}(x)]; \lambda_\beta = \|f_\beta\|, \mu_\beta = \|g_\beta\|$.

We assume that

$$1 \le p_0 \le p(x) \le P < \infty, |E_\infty| = 0. \tag{1.1}$$

and recall that

$$\|f\|_p = \|f\|_{(p)} = \inf \left\{ \lambda : \lambda > 0, \int_\Omega \left| \frac{f(x)}{\lambda} \right|_{p(x)} dx \ \le 1 \right\} \tag{1.2}$$

and

$$\int_\Omega \left| \frac{f(x)}{\|f\|_p} \right|^{p(x)} dx = 1, \|f\|_p \ne 0 \tag{1.3}$$

under the assumption (1.1).

1.1. The norming value and its bounds.

Similarly to a function $f_\beta(x)$ we denote

$$g_\beta = g_\beta(x) = g_\beta(x_0,r;x) \ := |x-x_0|^{\beta(x_0)} \bar{\chi}_{B(x_0,r)}(x) \tag{1.4}$$

where $\bar{\chi}_{B(x_0,r)}(x) = 1 - \chi_{B(x_0,r)}(x)$. Let

$$\mu_\beta = \mu_\beta(x_0,r) \ := \|g_\beta\|_p \ . \tag{1.5}$$

Under the asumptions (1.1), by (1.3) we have

$$\int_{|y|\ge r} \left(\frac{|y|^{\beta(x_0)}}{\mu_\beta} \right)^{p(x_0+y)} dy = 1. \tag{1.6}$$

Lemma 1.1. *The function* $\mu_\beta(x_0,r)$ *is increasing in* r *. If* $p(x) \in$ *w-Lip* (R^n) *and* $n + \beta(x_0)p(x_0) \le 0$, *then* $\lim_{r\to 0} \mu_\beta(x_0,r) = \infty$.

Proof is straightforward.

Definition 1.2. *The value* $r = r_0$ *is the norming value for the function (1.4) in the space* $L^{p(x)}(R^n)$ *, if* $\mu_\beta(x_0,r_0) = 1$.

Lemma 1.3. *The norming value* r_0 *is the root of the equation*

$$\int_{|x|>r_0} |x|^{\beta(x_0)p(x+x_0)} dx = 1. \tag{1.7}$$

This root does exist if $p(x) \in$ *w-Lip* $(\dot{R}^n)$ *and*

$$n + \beta(x_0)p(x_0) \leq 0 \ , \ n + \beta(x_0)p(\infty) < 0. \tag{1.8}$$

Lemma's assertion is an immediate consequence of the monotonicity of the left hand-side in (1.7) with respect to r_0 (cf. the proof of Lemma 2.11 in [1]).

Lemma 1.4. *Let* $p(x) \in$ *w-Lip* (R^n) *. If*

$$\sup_{x\in R_n} [n + \beta(x_0)p(x_0)] \ =: -d_0 < 0 \ , \ \sup_{x\in R_n} |\beta(x)| \ := \ B < \infty \tag{1.9}$$

then the function $r_0(x_0)$ *has a positive lower bound :*

$$r_0(x_0) \geq \left[1 + (n + BP)e^{AB}\right]^{-\frac{1}{d_0}} \tag{1.10}$$

where A *is the constant from the w-Lip-condition for the function* $p(x)$*.*

Proof. Assuming that $r_0(x_0) \leq 1$, from (1.7) we have

$$\int_{r_0<|x|<1} |x|^{\beta(x_0)p(x_0)} g(x)dx \leq 1, \quad g(x) = |x|^{\beta(x_0)[p(x+x_0)-p(x_0)]}. \tag{1.11}$$

Here $g(x) \geq e^{-AB}$ by (2.10) from [1]. Therefore, from (1.11) $\int_{r_0<|x|<1} |x|^{\beta(x_0)p(x_0)} dx \leq e^{AB}$. Hence $r_0^{-|n+\beta(x_0)p(x_0)|} - 1 \leq |n + \beta(x_0)p(x_0)| e^{AB}$, which implies (1.10). □

Lemma 1.5. *Let* $p(x)$ *be continuous in a neighbourhood of infinity. If* $\sup_{x\in R^n} \beta(x) < -\frac{n}{p(\infty)}$ *, then*

$$\sup_{x\in R^n} r_0(x) = c < \infty. \tag{1.12}$$

Proof. By continuity of $p(x)$ at infinity we conclude that there exists $R_0 > 0$ such that

$$\delta_{R_0} \ := \inf_{x\in R_n} |\beta(x)| \inf_{|\xi|\geq R_0} p(\xi) - n > 0. \tag{1.13}$$

Assuming that $r_0(x_0) \geq 1$, we derive from (1.7) the folowing

$$1 = \int_{|x|>r_0, |x+x_0|\geq R_0} |x|^{\beta(x_0)p(x+x_0)} dx \ + \ \int_{|x|>r_0, |x+x_0|\leq R_0} |x|^{\beta(x_0)p(x+x_0)} dx \leq$$

$$\int_{|x|>r_0} |x|^{-n-\delta_{R_0}} dx + \int_{|x+x_0|\leq R_0} r_0^{-|\beta(x_0)|p_0} dx.$$

Hence $1 \leq |S_{n-1}| \left(\int_{r_0}^{\infty} \rho^{-1-\delta_{R_0}} d\rho + \frac{R_0^n}{n} r_0^{-|\beta(x_0)|p_0} \right)$. Therefore,

$$1 \leq |S_{n-1}| \left(\delta_{R_0}^{-1} r_0^{-\delta_{R_0}} + \frac{R_0^n}{n} r_0^{-\frac{-np_0}{p(\infty)}} \right). \tag{1.14}$$

Hence (1.12) follows with $c = \left[|B_n| \left(\frac{n}{\delta_{R_0}} + R_0^n \right) \right]^{\frac{1}{\gamma}}$, $\gamma = \min \left(\delta_{R_0}, \frac{np_0}{p(\infty)} \right)$.

1.2. Estimates for the norm μ_β as $r \to 0$.

Before the main estimate in Theorem 1.8 , we give some "rough" estimates in Lemma 1.6 which will be used then in the proof of Theorem 1.8.

Together with (1.9) we shall need the condition

$$d_\infty \ := - \sup_{x \in R_n} [n + \beta(x_0)p(\infty)] > 0, \tag{1.15}$$

which in fact was used in Lemma 1.5.

Lemma 1.6. *Let $p(x) \in$ w-Lip $(\dot{R}^n)$. Under the conditions (1.9) there exists $c > 0$ not depending on r and x_0 such that*

$$\mu_\beta \leq c r^{\beta(x_0)} \ , \quad 0 < r < r_0.$$

Proof. From (1.6) we have

$$1 = \int_{r<|x|<r_0, |x|^\beta < \mu_\beta} \left(\frac{|x|^{\beta(x_0)}}{\mu_\beta} \right)^{p(x_0+x)} dx \ +$$

$$\int_{r<|x|<r_0, |x|^\beta > \mu_\beta} \left(\frac{|x|^{\beta(x_0)}}{\mu_\beta} \right)^{p(x_0+x)} dx \ + \ \int_{|x|>r_0} \left(\frac{|x|^{\beta(x_0)}}{\mu_\beta} \right)^{p(x_0+x)} dx$$

Hence, taking (1.7) into account, we obtain

$$1 \leq \int_{r<|x|<r_0} \left[\left(\frac{|x|^{\beta(x_0)}}{\mu_\beta} \right)^P + \left(\frac{|x|^{\beta(x_0)}}{\mu_\beta} \right)^{p_0} \right] dx + \frac{1}{\mu_\beta^{p_0}} \leq$$

$$|B(0,r_0)|\left[\left(\frac{r^{\beta(x_0)}}{\mu_\beta}\right)^P+\left(\frac{r^{\beta(x_0)}}{\mu_\beta}\right)^{p_0}\right]+\frac{1}{\mu_\beta^{p_0}}\ . \tag{1.16}$$

By Lemma 1.5, $|B(0,r_0)|$ is bounded uniformly in x_0 . Noting also that $\beta=\beta(x_0)<0$, we reduce (1.16) to $\left(\frac{r^{\beta(x_0)}}{\mu_\beta}\right)^P+\left(\frac{r^{\beta(x_0)}}{\mu_\beta}\right)^{p_0}\geq C$. Hence the estimate $\frac{r^{\beta(x_0)}}{\mu_\beta}\geq c_1>0$ evidently follows. □

Lemma 1.7. *Let* $a(x)\geq 0,\ p(x)\geq 1,\ x\in R^n,$ *and* $p(x)\in C(\dot{R}^n)$. *If*

$$a(x)p(x)\geq n+d_0,\quad d_0>0, \tag{1.17}$$

$$a(x)p(\infty)\geq n+d_\infty,\quad d_\infty>0, \tag{1.18}$$

then there exist numbers $N>0$ *and* $\epsilon>0$ *such that*

$$a(x)p(\xi)\geq n+d,\qquad d=\frac{1}{2}\min(d_0,d_\infty) \tag{1.19}$$

for all $x\in R^n$ *and* $\xi\in R^n$ *such that either* $|\xi|\geq N$ *or* $|x-\xi|\leq\epsilon$.

Proof is straightforward.

Theorem 1.8. *Let* $p(x)\in$ *w-Lip* $(\dot{R^n})$ *and let the conditions (1.9) and (1.15) be satisfied. Then*

$$\mu_\beta\leq cr^{\beta(x_0)+\frac{n}{p(x_0)}},\quad 0<r\leq r_0, \tag{1.20}$$

with $c>0$ *not depending on* r *and* x_0 .

Proof. Denoting $\mu=\mu_\beta$ for brevity, from (1.6) we have

$$1=\int_{r<|x|<\epsilon,\mu|x|^{|\beta|}<1}+\int_{r<|x|<\epsilon,\mu|x|^{|\beta|}>1}+\int_{|x|>\epsilon}=I_1+I_2+I_3 \tag{1.21}$$

where it is assumed that $r<\epsilon$, a fixed number $\epsilon\in(0,1]$ being chosen later.

E s t i m a t i o n o f I_1 . We represent I_1 as $I_1=\int_{r<|x|<\epsilon,\mu|x|^{|\beta|}<1}g_r(x)$ $(|x|^{|\beta|}\mu)^{-p(x_0)}dx,\quad g_r(x)=\left(\mu|x|^{|\beta|}\right)^{p(x_0)-p(x+x_0)}$. By the w-Lip-condition for $p(x)$ we have

$$|\log g_r(x)|\leq A\left|\frac{\log\left(|x|^{|\beta|}\mu\right)}{\log\frac{|x|}{2}}\right|=A\ \frac{|\beta|\log\frac{1}{|x|}-\log\mu}{\log\frac{2}{|x|}}\leq A|\beta|\leq AB.$$

Therefore, $I_1 \leq \frac{e^{AB}}{\mu^{p(x_0)}} \int_{|x|>r} |x|^{\beta(x_0)p(x_0)}dx$. Hence

$$I_1 \;\leq\; \frac{e^{AB}|S_{n-1}|}{d_0}\frac{r^{n+\beta(x_0)p(x_0)}}{\mu^{p(x_0)}} \tag{1.22}$$

E s t i m a t i o n of I_2 . We have

$$I_2 \leq \int_{|x|>r}\left(\frac{|x|^{\beta(x_0)}}{\mu}\right)^{p_\epsilon} dx \tag{1.23}$$

where $p_\epsilon = \min_{|x-x_0|<\epsilon} p(x)$. Now we choose ϵ independent of x_0 sufficiently small so that $n + \beta(x_0)p_\epsilon \leq -\delta < 0$ with δ also independent of x_0 . Such a choice is possible by Lemma 1.6. Then from (1.23) we have

$$I_2 \leq \frac{|S_{n-1}|}{\mu^{p_\epsilon}}\int_r^\infty \rho^{\beta(x_0)p_\epsilon+n-1}d\rho = \frac{|S_{n-1}|}{\delta}\frac{r^{n+\beta(x_0)p_\epsilon}}{\mu^{p_\epsilon}}. \tag{1.24}$$

E s t i m a t i o n o f I_3. Evidently,

$$I_3 \leq \frac{I_0}{\mu^{p_0}}, \quad I_0 := \int_{|x|>\epsilon}|x|^{\beta(x_0)p(x+x_0)}dx \tag{1.25}$$

Let us show that I_0 is bounded , ϵ being fixed. If $r_0 \leq \epsilon$, then $I_0 \leq 1$ by (1.6). So, let $r_0 > \epsilon$. Using (1.6) again, we have

$$I_0 = 1 + \int_{\epsilon<|x|<r_0}|x|^{\beta(x_0)p(x+x_0)}dx \leq 1 + \epsilon^{\beta(x_0)P}\frac{|S_{n-1}|}{n}r_0^n \leq 1 + \epsilon^{-BP}|B_n|r_0^n.$$

It remains to note that r_0^n is bounded by Lemma 1.5. Then, in view of boundedness of I_0 the inequality (1.25) implies the estimate $I_3 \leq C\mu^{-p_0}$.

Gathering the estimates for I_1, I_2, I_3, we conclude from (1.21) that

$$1 \leq c_0\left[\frac{r^{n+\beta(x_0)p(x_0)}}{\mu^{p(x_0)}} + \frac{r^{n+\beta(x_0)p_\epsilon}}{\mu^{p_\epsilon}} + \frac{1}{\mu^{p_0}}\right]. \tag{1.26}$$

Evidently, $\mu(r)$ increases to infinity as $r \to 0$. Therefore, $\frac{1}{\mu^{p_0}} \leq \frac{1}{2c_0}$ for r sufficiently small. Then from (1.26) we derive the inequality

$$\frac{r^{n+\beta(x_0)p(x_0)}}{\mu^{p(x_0)}} + \frac{r^{n+\beta(x_0)p_\epsilon}}{\mu^{p_\epsilon}} \geq \frac{c_0}{2}\ . \tag{1.27}$$

Here $\frac{r^{n+\beta(x_0)p_\epsilon}}{\mu^{p_\epsilon}} \leq c_1\frac{r^{n+\beta(x_0)p(x_0)}}{\mu^{p(x_0)}}, \quad c_1 = \max(1, c^{P-p_0})$, by Lemma 1.6. Then from (1.27) it follows that $\frac{r^{n+\beta(x_0)p(x_0)}}{\mu^{p(x_0)}} \geq \frac{c_0}{2(1+c_1)}$ which yields (1.20). □

1.3. Estimates of the norms μ_β as $r \to \infty$.

It is natural to expect that instead of (1.20) we should have

$$\mu_\beta \leq C r^{\beta(x_0)+\frac{n}{p(\infty)}}, \quad r \to \infty \ . \tag{1.28}$$

However, it proved to be a difficult moment and we succeeded only in obtaining such an estimate with a constant c which depends on x_0 and grows as $|x_0| \to \infty$ (as a power function of $|x_0|$). It is also possible to obtain the estimate (1.28) with an absolute constant but with a worse exponent.

A) Estimates with constants depending on the point x_0.

We start with the following auxilliary lemma giving the rough lower bound which nevertheless proved to be of importance in proving the main result.

Lemma 1.9. *Let $p(x) \in w$-Lip $(\dot{R}^n)$ and let (1.15) and the second of the conditions (1.9) be satisfied. Then $\mu_\beta \ \geq \ 2^{-\frac{B}{n}} r^{\beta(x_0)}$ for $r \geq |B_n|^{-1/n}$.*

Proof. We assume that $\mu = \mu_\beta \leq r^\beta$, otherwise the lemma is proved. From (1.6) we have

$$1 \geq \int_{r<|y|<\mu^{1/\beta}} \left(\frac{|y|^{\beta(x_0)}}{\mu_\beta}\right)^{p(x_0+y)} dy \geq \int_{r<|y|<\mu^{1/\beta}} dy = |B_n| \left(\mu^{\frac{n}{\beta}} - r^n\right).$$

Then $\mu_\beta^{\frac{n}{\beta}} \ \leq \ \frac{1}{|B_n|} + r^n \leq 2r^n$ in case of $r \geq |B_n|^{-1/n}$. . Hence the lemma's assertion follows. □

Theorem 1.10. *Under the assumptions of Lemma 1.9*

$$\frac{c_1}{K} \ r^{\beta(x_0)+\frac{n}{p(\infty)}} \ \leq \ \mu_\beta \ \leq \ c_2 K \ r^{\beta(x_0)+\frac{n}{p(\infty)}} \ , \tag{1.29}$$

as $r \to \infty$ ($r \geq \left[\max(2, \frac{1}{|B_n|}\right]^{1/n}$) where c_1, c_2 do not depend on r and x_0 , while K, not depending on r, may grow when $|x_0| \to \infty$.

Proof. For $\mu = \mu_\beta$ from (1.6) we have

$$1 = \int_{|x|>r} \left(\frac{|x|^\beta}{\mu}\right)^{p(\infty)} j_r(x)dx \tag{1.30}$$

where $j_r(x) \ = \ \left(\frac{|x|^\beta}{\mu}\right)^{p(x+x_0)-p(\infty)} \ \leq \ 2^{\frac{B(P-P_0)}{n}} \left(\frac{|x|^\beta}{2^{B/n}\mu}\right)^{p(x+x_0)-p(\infty)}$. Therefore, $\log j_r(x) \leq \log C + |p(x+x_0) - p(\infty)| \left(|\beta(x_0)| \log|x| + \frac{B}{n}\log 2 - \log\mu\right)$.

with $C = 2^{\frac{\beta(P-p_0)}{n}}$. Taking into account the w-Lip - behavior of the function $p(x)$ at infinity, we obtain $|\log j_r(x)| \leq \log C + A_\infty B \frac{\log|x|+\frac{\log 2}{n}}{\log(e+|x+x_0|})) \leq \log C + 2A_\infty B \frac{2\log|x|}{\log(e+|x+x_0|)}$. Since $\frac{\log|x|}{\log|y|} \leq \frac{\log(a+|x-y|}{\log|a|}$ for $|\ y\ | \geq a > 1$, we have $|\log j_r(x)| \leq \log C + 2A_\infty B\log(e+|x+x_0|) \leq$ and then $\frac{1}{c}(e+|x_0|)^{-2A_\infty B} \leq j_r(x) \leq c(e+|x_0|)^{2A_\infty B}$. So, from (1.30)

$$\frac{1}{c}(e+|x_0|)^{-2A_\infty B}\int_{|x|>r}\left(\frac{|x|^\beta}{\mu}\right)^{p(\infty)} dx \leq 1 \leq c(e+|x_0|)^{2A_\infty B}\int_{|x|>r}\left(\frac{|x|^\beta}{\mu}\right)^{p(\infty)} dx.$$

Simple evaluations reduce this to (1.29) with $K = (e+|x_0|)^{2A_\infty B/p(\infty)}$.□

B) Estimates with absolute constants but with rough exponents.

Lemma 1.11. *Let $r \geq r_0$. Then*

$$[J_\beta(x_0,r)]^{\frac{1}{p_r}} \leq \mu_\beta \leq [J_\beta(x_0,r)]^{\frac{1}{P_r}} \tag{1.31}$$

where

$$J_\beta(x_0,r) = \int_{|x|>r} |x|^{\beta(x_0)p(x+x_0)}dx. \tag{1.32}$$

Proof. Since $\mu_\beta \geq 1$ if and only if $0 < r < R_0$, the equality (1.6) gives

$$\frac{J_\beta(x_0,r)}{\mu^{p_r}} \ \leq\ 1\ \leq \frac{J_\beta(x_0,r)}{\mu^{P_r}}$$

which coincides with (1.31). □

Lemma 1.12. *For $r \geq r_0$ the integral (1.32) is estimated as follows*

$$k(x_0,r)\left(\frac{r}{r_0}\right)^{n+\beta(x_0)P_r} \ \leq\ J_\beta(x_0,r)\ \leq\ k(x_0,r)\left(\frac{r}{r_0}\right)^{n+\beta(x_0)p_r} \tag{1.33}$$

where

$$k(x_0,r) = \int_{|x|>r_0} |x|^{\beta(x_0)p(\frac{r}{r_0}x+x_0)}dx \tag{1.34}$$

Proof. The change of variables $x = \frac{r}{r_0}y$ in (1.32) leads to

$$J_\beta(x_0,r) = \left(\frac{r}{r_0}\right)^n \int_{|y|>r_0}\left(\frac{r}{r_0}|y|\right)^{\beta(x_0)p(\frac{r}{r_0}y+x_0)} dy\ .$$

Hence (1.33) follows. □

Lemma 1.13. *Let $p(x) \in$ w-Lip (R^n) and let the assumptions (1.9) and (1.15) be satisfied. Then $0 < c_1 \le k(x_0, r) \le c_2 < \infty$ for $r \ge r_0$ with c_1 and c_2 not depending on r and x_0.*

Proof. By (1.15) , there exists $N > 0$ such that

$$\inf_{x\in R^n} |\beta(x)| \inf_{|\xi|\ge N} p(\xi) \ge n + \frac{d_\infty}{2} \qquad (1.35)$$

(see Lemma 1.7). We have $k(x_0, r) = \int_{|x|>r_0,|\xi|\ge N} + \int_{|x|>r_0,|\xi|<N}$ where $\xi = \frac{r}{r_0}x + x_0$. Let $r_0 \ge 1$ first. By (1.35) we have $k(x_0, r) \le \int_{|x|>r_0} |x|^{-n-\frac{d_\infty}{2}} dx + \int_{|\xi|l<N} r_0^{-|\beta(x_0)|p_0} dx = |S_{n-1}| \left[\frac{2}{d_\infty} r_0^{-\frac{d_\infty}{2}} + \frac{1}{n}\, r_0^{-|\beta(x_0)p_0} \left(\frac{r_0 N}{r}\right)^n \right]$. It remains to note that $\frac{r_0}{r} \le 1$ and r_0 has a lower bound by Lemma 1.4.

If $r_0 < 1$, to obtain the right-hand side estimate, we put $k(x_0, r) = \int_{|x|>1,|\xi|\ge N} + \int_{|x|>1,|\xi|<N} + \int_{r_0\le|x|\le 1}$. Here the first two integrals are estimated exactly as before, while for the third one we have $\int_{r_0\le|x|\le 1} \le \int_{c\le|x|\le 1} |x|^{-BP} dx = const < \infty$ where $c = \inf_{x\in R^n} r_0(x) > 0$ by Lemma 1.4.

It remains to give the lower bound for $k(x_0, r)$ which is easier :

$$k(x_0, r) \ge \int_{|x|\ge\max(1,r_0)} |x|^{-BP} dx \ge \int_{|x|\ge\max(1,c)} |x|^{-BP} dx = const > 0. \square$$

Theorem 1.14. *Let $p(x) \in$ w-Lip (R^n) and let the assumptions (1.9) and (1.15) be satisfied. Then*

$$c_1 \left(\frac{r}{r_0}\right)^{\frac{n+\beta(x_0)\bar{P}_r}{p_r}} \le \mu_\beta \le c_2 \left(\frac{r}{r_0}\right)^{\frac{n+\beta(x_0)\bar{p}_r}{P_r}} , r \ge r_0 , \qquad (1.36)$$

with $c_1 > 0$ and $c_2 > 0$ not depending on r and x_0.

Proof. The estimates (1.36) follow directly from Lemmas 1.11 - 1.13.

Remark 1.15. *Evidently, $\lim_{r\to\infty} \bar{p}_r = \lim_{r\to\infty} \bar{P}_r = p(\infty)$, if $p(x)$ is continuous at infinity. However, these limits are in no way uniform in x_0*

1.4. The case of a bounded domain.

In case of spaces $L^{p(x)}(\Omega)$ on a bounded domain $\Omega \subset R^n$, Theorem 1.8 on behaviour of norms of power functions of distance remains true with the simplification owing to the fact that now there is no necessity to deal with the assumption (1.15) connected with infinity. Naturally, the function (1.4) is considered in the domain Ω only with the corresponding integration in (1.6).

Definition 1.16. *We say that* $f(x) \in$ *w-Lip* $(\bar{\Omega})$ *if* $f(x) \in C(\bar{\Omega})$ *and* $|f(x)-f(y)| \log \frac{1}{|x-y|} \leq A < \infty$ *for all* $x, y \in \bar{\Omega}$ *such that* $|x-y| < 1$.

Analysis of the proof of Theorem 1.8 shows that the following theorem holds.

Theorem 1.17. *Let* $f(x) \in$ *w-Lip* $(\bar{\Omega})$ *and let* $\sup_{x \in \bar{\Omega}} [n + \beta(x)p(x)] < 0$, $\sup_{x \in \bar{\Omega}} |\beta(x)| < \infty$. *Then*

$$\mu_\beta(x_0, r) \leq c r^{\beta(x_0) + \frac{n}{p(x_0)}} \ , \ x_0 \in \Omega \ , \ 0 \leq r \leq D \ , \tag{1.37}$$

where $D = diam\ \Omega < \infty$ *and* c *does not depend on* x_0 *and* r.

II. Convolution operators in the spaces $L^{p(x)}$

2.1. On a necessary condition for the Young theorem in case of kernels of potential type.

We start with the folowing remark of a negative character.

Remark 2.1. *Young theorem in the form*

$$\|Kf\|_p \ \leq \ c \|k\|_1 \|f\|_p \tag{2.1}$$

is not valid for an arbitrary kernel $k(x) \in L^1(R^n)$ *and an arbitrary variable exponent* $p(x), 1 \leq p(x) \leq \infty$.

Proof. Let $n = 1$ for simplicity. We put $p(x) = p_1$ for $x < 0$ and $p(x) = p_2$ for $x > 0$ where $1 \leq p_1 < p_2 < \infty$ and $k(x) = |x-2|^{\alpha-1}$ if $|x| \leq 3$ and $k(x) = 0$ if $|x| > 3$ where $0 < \alpha < \frac{1}{p_1} - \frac{1}{p_2}$, so that $k(x) \in L^1(R^1)$. Then we choose $f(x) = |x+1|^{-\nu}$ if $x \in (-2, 0)$ and $f(x) = 0$ otherwise. Evidently, $f(x) \in L^{p(x)}(R^1)$ if we take $0 < \nu < \frac{1}{p_1}$. However, $k * f$ does not belong to $L^{p(x)}(R^1)$ under the additional choice $\nu > \alpha + \frac{1}{p_2}$, the latter choice being evidently possible. Really, taking $1 < x < \frac{3}{2}$, we have

$$k * f \geq \int_{x-3}^{-1} |x-t-2|^{\alpha-1} |t+1|^{-\nu} dt = \int_0^{2-x} s^{-\nu} (x-1+s)^{\alpha-1} ds =$$

$$(x-1)^{\alpha-\nu}\int_0^{\frac{2-x}{x-1}}\xi^{-\nu}(1+\xi)^{\alpha-1}d\xi \ \geq \ \frac{c}{(x-1)^{\nu-\alpha}}$$

with $c=\int_0^1\xi^{-\nu}(1+\xi)^{\alpha-1}d\xi$. Therefore , $k*f$ cannot be integrable on $[1,\frac{3}{2}]$ to the power p_2 since $(\nu-\alpha)p_2>1$.□

This example shows clearly that convoluted functions $k(x)$ and $f(x)$ having singularities at points a and x_0, respectively, produce the convolution with a singularity at the point x_0+a (see also Lemma 2.3 below). In case of a constant exponent p such a shift of singularities played no role. However, for a variable exponent $p(x)$ having, in general, different values at points x_0 and x_0+a, such shifting produces evident problems. So, we begin with kernels $k(x)$ having singularities only at the origin, potential type operators, for example. Nevertheless, we consider before some "potential type operators" with weak singularities at different points and derive some necessary condition for them to act within the frameworks of the spaces $L^{p(x)}(R^n)$.

Let $E=\{x_1,...,x_N\}$ be a finite set of points in R^n and let $\alpha=(\alpha_1,...,\alpha_N)\in R_+^N$.

Definition 2.2. *By $A=A_{E,\alpha}$ we denote the class of kernels which satisfy the assumptions*

1) $k(x)\geq 0$;

2)$k(x)\geq c|x-x_k|^{\alpha_k-n},\ 0<\alpha_k\leq n, c>0;\ |x-x_k|<\epsilon, k=1,...,N,$ for some $\epsilon>0$;

3) $k(x)$ is bounded beyond some neighbourhoods of the points $x_1,...,x_N$.

Lemma 2.3. *Let $k(x)\in A_{E,\alpha}$ and let $f(x)=|x-a|^{-\gamma(x)}$ for $|x-a|\leq d$ and $f(x)=0$ for $|x-a|>d$ where $a\in R^n, d>0, \gamma(x)\in$ w-Lip $(B(a,d))$ and $0<\gamma(a)<n$. Then the convolution $k*f$ has singularities at the points $a+x_1, a+x_2, ..., a+x_N$:*

$$k*f\geq c|x-(a+x_k)|^{\alpha_k-\gamma(a)}\ ,\ |x-(a+x_k)|\leq\min(\delta,\frac{\epsilon}{2}) \qquad (2.2)$$

where $c>0$ and ϵ is the number from Definition 2.2.

Proof. We consider the convolution

$$k*f\geq\int_{|y-a|<\delta}\frac{k(x-y)dy}{|y-a|^{\gamma(y)}},\quad 0<\delta\leq d\ ,$$

for $x\in\left\{x:|x-x_k-a|<\frac{\epsilon}{2}\right\}$. Choosing $\delta\leq\frac{\epsilon}{2}$, we have $|x-y-x_k|<\epsilon$.

Then, by the condition 2) of Definition 2.2, we obtain

$$k * f \geq \int_{|y-a|<\delta, |x-y-x_k|<\epsilon} |y-a|^{-\gamma(y)} |x-y-x_k|^{\alpha_k - n} dy \ .$$

Since $\gamma(x) \in$ w-Lip , we have $0 < m \leq |x-a|^{\gamma(x)-\gamma(a)} \leq M < \infty$. So, $k*f \geq c_1 \int_{|t|<\delta, t-\tilde{x}|<\epsilon} |t|^{-\gamma(a)} |t-\tilde{x}|^{\alpha_k - n} dt$ where we have denoted $\tilde{x} = x - x_k - a$. Let $r = |\tilde{x}|$. After the change of variables $t = r\xi$ we obtain

$$k * f \geq c r^{\alpha_k - \gamma(a)} \int_{|\xi|<\delta/r, |\xi - r^{-1}\tilde{x}|<\epsilon r^{-1}} |\xi|^{-\gamma(a)} |\xi - \frac{\tilde{x}}{r}|^{\alpha_k - n} d\xi.$$

Applying also an evident rotation change of variables we arrive at the inequality

$$k * f \geq A(r) r^{\alpha_k - \gamma(a)} \tag{2.3}$$

with $A(r) = \int_{|\xi|<\delta/r, |\xi - \bar{e}|<\epsilon r^{-1}} |\xi|^{-\gamma(a)} |\xi - |^{\alpha_k - n} d\xi$. and $\bar{e} = (1, 0, ..., 0)$. The required inequality (2.2) will follow from (2.3) if $A(r)$ does not vanish for small r. The latter is easily seen from the fact that $\left\{\xi : |\xi| < \frac{\delta}{r}, |\xi - e| < \frac{\epsilon}{r}\right\} \supseteq \{|\xi| < 1\}$ for $r \leq \min(\delta, \epsilon/2)$.□

Lemma 2.3 yields the following more essential statement.

Lemma 2.4. *Let $k(x) \in A_{E,\alpha}$, $p(x) \in$ w-Lip $(R^n), 1 \leq p(x) \leq P < \infty$. If the convolution operator (1) maps the space $L^{p(x)}(R^n)$ into itself, then necessarily*

$$\frac{1}{p(x)} - \frac{1}{p(x+x_k)} \leq \frac{\alpha_k}{n} \ , k = 1, 2, ..., N, \tag{2.4}$$

for all $x \in R^n$.

Proof. Suppose that $k * f \in L^{p(x)}(R^n)$ for any function $f(x) \in L^{p(x)}(R^n)$. We choose then $f(x) = |x-a|^{-\frac{n-\delta}{p(x)}} \chi_{B(a,d)}(x)$ where $\chi_{B(a,d)}(x)$ is the characteristic function of an arbitrary ball $B(a,d), a \in R^n, 0 < d < \infty$, and $0 < \delta < n$. From Lemma 2.3 it follows that $|x - (a + x_k)|^{\alpha_k - \frac{n-\delta}{p(a)}} \in L^{p(x)}$ in some neighbourhood of the point $a + x_k, \ k = 1, 2, ..., N$. Then, by Lemma 2.7 from [1], the necessary condition

$$\left[\frac{n-\delta}{p(a)} - \alpha_k\right] p(a + x_k) < n$$

should be satisfied, which coincides with (2.4). □

Remark 2.5. *The condition (2.4) is quite natural in the following sense: it means that in case of potential type kernels with a singularity shifted from the origin to a point x_k, the value of the exponent $p(x)$ at the new (shifted) point $x + x_k$ should not be greater that the value of the corresponding Sobolev exponent*

$$\frac{np(x)}{n - \alpha_k p(x)}$$

(in case $p(x) < n/\alpha_k$), calculated with respect to the "old" point x.

Remark 2.6. *The condition (2.4) is satisfied automatically in the following cases:*

1) in case of purely potential kernel, that is $E = \{0\}$, when there is no shift of singularities ;

2) in case of the exponent $p(x)$ which is "periodically nonincreasing" in each of the directions defined by vectors $x_k \in E$, that is $p(x + x_k) \leq p(x), \quad x \in R^n, \quad x_k \in E$ (in particular, if $p(x)$ is periodic with respect to all vectors in E).

2.2. Young theorem .

The above arguments show that it is impossible to have the Young theorem, except for special cases noted e.g. in Remark 6. Within the framework of the assumption $k(x) \in L^1(R^n)$, or, more generally, $k(x) \in L^{q(x)}(R^n)$, it proves to be possible to obtain a Young-type theorem in terms of the upper and lower bounds for $p(x)$ and $q(x)$. One of such versions of Young theorem is considered below. We prove first Young theorem in a special form, when the resulting exponent is constant (Theorem 2.7) and then derive more general statement (Theorem 2.8). Everywhere below, as before

$$p_0 = \inf_{x \in R^n} p(x) \ , \quad P = \sup_{x \in R^n} p(x), \ 1 \leq p_0 \leq P < \infty,$$

$$q_0 = \inf_{x \in R^n} q(x) \ , \quad Q = \sup_{x \in R^n} q(x), \ 1 \leq q_0 \leq Q < \infty,$$

Theorem 2.7. *Let $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1 + \frac{1}{r}$ where $r = const \geq 1$. If $k(x) \in L^{q_0}(R^n) \bigcap L^Q(R^n)$, then the convolution operator (1) is bounded from $L^{p(x)}(R^n)$ into $L^r(R^n)$.*

Proof. We assume that $\|f\|_p \leq 1$. Evidently,

$$|Kf(x)| \leq \int_{R^n} A^{1-\mu(y)} |f(y)|^{\frac{p(y)}{r}} \, |k(x-y)|^{\mu(y)} \, |f(y)|^{1-\frac{p(y)}{r}} \left| \frac{k(x-y)}{A} \right|^{1-\mu(y)} dy$$

where $A>0$ and $\mu(y)$, $0<\mu(y)<1$, will be chosen later.

Applying the Holder inequality (1.21′) from [1] , with the exponents $p_1(y)=r,\ \ p_2(y)=\frac{rp(y)}{r-p(y)},\ \ p_3(y)=p\prime(y)=\frac{p(y)}{p(y)-1}$, we obtain

$$|Kf(x)|\le c\left\{\int_{R^n}A^{r-r\mu(y)}|f(y)|^{p(y)}\,|k(x-y)|^{r\mu(y)}\,dy\right\}^{\frac{1}{r}}$$

$$\times\left\||f(y)|^{1-\frac{p(y)}{r}}\right\|_{p_2(y)}\left\|\left|\frac{k(x-y)}{A}\right|^{1-\mu(y)}\right\|_{p\prime(y)} \tag{2.5}$$

In view of the estimate (1.16) from [1] we obtain

$$\left\||f(y)|^{1-\frac{p(y)}{r}}\right\|_{p_2(y)}\le\|f\|_{p(y)}^{\min[1-\frac{p(y)}{r}]}\le 1. \tag{2.6}$$

To estimate the third factor in (2.5) we choose $\mu(y)$ in such a way that $[1-\mu(y)]\,p\prime(y)=q(y)$, that is $\mu(y)=\frac{q(y)}{r}$ We intend to apply the inequality (1.16) from [1] to this third factor. So, we need the inequality

$$\left\|\frac{k(x-y)}{A}\right\|_{q(y)}=\frac{1}{A}\,\|k(x-y)\|_{q(y)}\le\ 1. \tag{2.7}$$

To reach (2.7) we choose $A=\|k\|_{q_0}+\|k\|_Q$ so that (2.7) is satisfied by Lemma 1.13 from [1] . Then we can apply (1.16) from [1] and obtain

$$\left\|\left|\frac{k(x-y)}{A}\right|^{1-\mu(y)}\right\|_{p\prime(y)}\le 1. \tag{2.8}$$

By (2.6) and (2.8) we obtain from (2.5)

$$||Kf||_r\le cA^\nu\left\{\int_{R^n}dx\int_{R^n}|f(y)|^{p(y)}|k(x-y)|^{q(y)}dy\right\}^{\frac{1}{r}}=$$

$$cA^\nu\left\{\int_{R^n}|f(y)|^{p(y)}dy\int_{R^n}|k(x)|^{q(x+y)}dx\right\}^{\frac{1}{r}}$$

where $\nu=1-Q/r$ if $A\le 1$ and $\nu=1-q_0/r$ if $A\ge 1$. Hence, obviously

$$||Kf||_r\le cA^\nu\left(\|k\|_{q_0}^{\frac{q_0}{r}}+\|k\|_Q^{\frac{Q}{r}}\right)\int_{R^n}|f(y)|^{p(y)}dy$$

and it remains to note that the last integral is not greater than 1 according to (1.11) from [1] . □

Theorem 2.8. *Let* $k(x) \in L^{q_1}(R^n) \bigcap L^{q_2}(R^n), 1 \le q_1 \le q_2 < \infty$. *The convolution operator (1) is bounded from* $L^{p(x)}(R^n)$ *into* $L^{r(x)}(R^n)$, *if*

$$\frac{1}{q_1} - \frac{1}{q_2} \ge \frac{1}{p_0} - \frac{1}{P} \tag{2.9}$$

and $r(x)$ *is any bounded function,* $r(x) \ge 1$, *such that*

$$\frac{1}{p_0} + \frac{1}{q_2} - 1 \le \frac{1}{r(x)} \le \frac{1}{P} + \frac{1}{q_1} - 1. \tag{2.10}$$

Proof. In fact, Theorem 2.8 is a corollary of Theorem 2.7. Really, in the estimate for $\|Kf\|_r$ in Theorem 7 not the function $q(x)$ is involved, but only its upper and lower bounds Q and q_0 . Let us reconsider the assumption

$$\frac{1}{p(x)} + \frac{1}{q(x)} - 1 \equiv \frac{1}{r}$$

of Theorem 2.7 in terms of Q and q_0 . Since $\frac{1}{Q} \le \frac{1}{q(x)} \le \frac{1}{q_0}$, we have $\frac{1}{Q} \le \frac{1}{r} - \frac{1}{p(x)} + 1 \le \frac{1}{q_0}$. Hence $\frac{1}{r} + 1 - \frac{1}{q_0} \le \frac{1}{p(x)} \le \frac{1}{r} + 1 - \frac{1}{Q}$ for all $x \in R^n$. This is equivalent to $\frac{1}{r} + 1 - \frac{1}{q_0} \le \frac{1}{P}$, $\frac{1}{p_0} \le \frac{1}{r} + 1 - \frac{1}{Q}$. Hence

$$\frac{1}{r_2} := \frac{1}{p_0} + \frac{1}{Q} - 1 \le \frac{1}{r} \le \frac{1}{P} + \frac{1}{q_0} - 1 =: \frac{1}{r_1}.$$

By Lemma 1.13 from [1], $\|f\|_{r(x)} \le \|f\|_{r_1} + \|f\|_{r_2}$ for any function $r(x)$ such that $r_1 \le r(x) \le r_2$. Therefore, we arrive at the assertion of Theorem 2.8. □

If we deal with the convolution type operator in case of bounded domain:

$$K_\Omega f = \int_\Omega k(x-y)f(y)dy , \tag{2.11}$$

Theorem 8 is valid under weaker assumptions. Namely, the following its version is valid.

Theorem 2.9 . *Let* $k(x) \in L^Q((B(0, 2D))$ *where* $Q \ge 1$ *and* $D =$ *diam* Ω. *The operator (12) is bounded from* $L^{p(x)}(\Omega)$ *into* $L^{r(x)}(\Omega), r(x) \ge 1$, *if*

$$\frac{1}{Q} \le 1 - \frac{1}{p_0} + \frac{1}{P} , \quad \frac{1}{r(x)} \ge \frac{1}{Q} + \frac{1}{p_0} - 1.$$

Proof. Since $L^Q = L^Q \bigcap L^1$ in case of a set of finite measure, we may apply Theorem 2.8 with $q_1 = 1, q_2 = Q$, preliminarily having continued the function f and the kernel k as zero to the whole space and taking into account that the right-hand side inequality in (2.10) turns into trivial one. □

III. Potential type operators in $L^{p(x)}$

3.1. Sobolev Theorem.

We consider the potential type operator of variable order

$$I^{\alpha(x)} f = \int_\Omega \frac{f(y)dy}{|x-y|^{n-\alpha(x)}} \ , \quad x \in \Omega, \quad 0 < \alpha(x) < n, \tag{3.1}$$

where $\Omega \subseteq R^n$ is a bounded domain . We shall investigate mapping properties of the operator $I^{\alpha(x)}$ If within the framework of the spaces $L^{p(x)}(\Omega)$ (Sobolev theorem or its weaker version with pre-Sobolev exponent).

One can also consider the operator (3.1) with $\alpha(y)$ instead of $\alpha(x)$ which differs unessentially from (3.1) if $\alpha(x) \in$ w-Lip (Ω), because $c_1|x-y|^{n-\alpha(y)} \leq |x-y|^{n-\alpha(x)} \leq c_2|x-y|^{n-\alpha(y)}$ in this case.

We shall show that the Sobolev theorem in the natural generalization

$$I^{\alpha(x)} \quad : \quad L^p \to L^{q(x)}, \qquad \frac{1}{q(x)} = \frac{1}{p} - \frac{\alpha(x)}{n}, \quad \alpha(x) < \frac{n}{p} \tag{3.2}$$

is valid in case of constant $p > 1$. In case of variable $p(x)$ we succeeded in obtaining only the conventional result as yet: Sobolev theorem is valid if the maximal operator is bounded in the space $L^{p(x)}$. The question of boundedness of the maximal operator in $L^{p(x)}$ remains open.

For variable exponents $p(x)$ we shall also give an unconditional, but a weaker assertion about validity of such a theorem "with a gap" (Theorem 3.3). We specially treat the case of boundedness $I^{\alpha(x)} : L^{p(x)} \to L^r$ with the pre-limiting exponent r in the case when r is constant (Theorem 3.5).

Let

$$Mf(x) = \sup_{t>0} \frac{1}{t^n} \int_{y\in\Omega:|x-y|<t} |f(y)|dy, \quad x \in \Omega, \tag{3.3}$$

be the well known [2] maximal operator.

Definition 3.1. *A function $p(x), 1 < p(x) < \infty$, is said to be an appropriate function for the operator (3.3) if this operator is bounded in the space $L^{p(x)}(\Omega)$.*

It is well known that the constant function $p(x) \equiv p > 1$ is an appropriate function for the operator M . The question of existence of a non-trivial (non-constant) appropriate functions remains open. Supposingly, such a function $p(x)$ must be continuous. (In any case , Sobolev theorem cannot be valid for a piece-wise constant function $p(x)$, see Subsection 3.4 below, and therefore, the maximal operator cannot be bounded in such a space $L^{p(x)}$).

Theorem 3.2. (Sobolev type Theorem). *Let $1 < p < \infty$ and let*

$$\alpha_0 \ : = \inf_{x \in \Omega} \alpha(x) > 0 \ , \tag{3.4}$$

$$\sup_{x \in \Omega} \alpha(x) < \frac{n}{p} \ . \tag{3.5}$$

Then (3.2) holds . It remains valid for a variable exponent $p(x) \in w-Lip(\bar{\Omega}), 1 < p_0 \le p(x) \le P < \infty$ if $p(x)$ is an appropriate function for the maximal operator M and the assumption (3.5) is replaced by

$$\sup_{x \in \Omega} p(x)\alpha(x) < n. \tag{3.6}$$

Proof. We shall prove the second part of the theorem which assumes that the function $p(x)$ is appropriate for the maximal opertator. The first part is an immediate consequence of the second one. We shall use the known idea to reduce the Sobolev theorem to boundedness of maximal operator and the estimates for $L^{p(x)}$ - norms of power functions of distance we obtained before in [1] . We have

$$I^{\alpha(x)} f = \int_{|x-y|<r} |x-y|^{\alpha(x)-n} f(y)dy +$$

$$\int_{|x-y|>r} |x-y|^{\alpha(x)-n} f(y)dy \ = : A_r(x) + B_r(x) \tag{3.7}$$

so that $B_r(x) \equiv 0$ for $r \ge D = diam \ \Omega$.

We shall take use of the inequality

$$|A_r(x)| \ \le \ \frac{2^n r^{\alpha(x)}}{2^{\alpha(x)}-1} \ \ Mf(x) \tag{3.8}$$

which is known in case of $\alpha(x) = const$ and remains valid in case it is variable. By (3.8) and (3.4)

$$|A_r(x)| \le c_1 r^{\alpha(x)} Mf(x) \tag{3.9}$$

with some absolute constant $c_1 > 0$.

We assume that $\|f\|_{p(y)} \le 1$. Applying the Holder inequality (1.21′) from [1] to the integral $B_r(x)$, we obtain

$$|B_r(x)| \le \|f\|_{p(y)} \mu_\beta(x,r) \le \mu_\beta(x,r) \tag{3.10}$$

where

$$\mu_\beta(x,r) = \left\| |x-y|^{\beta(x)} \chi \right\|_{p\prime(y)} \tag{3.11}$$

and χ is the characteristic function of the exterior $\{y \in \Omega : |x-y| > r\}$ of the ball and $\beta(x) = \alpha(x) - n$. We can apply Theorem 1.17, its assumptions being satisfied due to conditions of our theorem. By that theorem we have $\mu_\beta(x,r) \le c_2 r^{-\frac{n}{q(x)}}, \quad x \in \Omega, \quad 0 < r < D$. If $p(x) \equiv p = const$, this estimate is evidently valid for all $r > 0$. Then from (3.7), in view of (3.9)-(3.11) , we obtain

$$\left| I^{\alpha(x)} f(x) \right| \le c_3 \left[r^{\alpha(x)} Mf(x) + r^{-\frac{n}{q(x)}} \right], \quad 0 < r < \infty,$$

Minimizing the right-hand side with respect to r we see that its minimum is reached at

$$r_{\min} = \left[\frac{1}{n} q(x) \alpha(x) Mf(x) \right]^{\frac{p(x)}{n}}$$

and easy evaluations give

$$\left| I^{\alpha(x)} f(x) \right| \le \frac{c_3 n}{p(x)} \left[\frac{q(x)}{n} Mf(x) \right]^{\frac{p(x)}{q(x)}} \left[\frac{1}{\alpha(x)} \right]^{1-\frac{p(x)}{q(x)}} .$$

Hence, by the assumptions of our theorem, $\left| I^{\alpha(x)} f(x) \right| \le c_4 \left[Mf(x) \right]^{\frac{p(x)}{q(x)}}$ so that

$$\left\| I^{\alpha(x)} f(x) \right\|_{q(x)} \le c_4 \left\| \left(Mf(x) \right)^{\frac{p(x)}{q(x)}} \right\|_{q(x)} .$$

Applying Lemma 1.12 from [1] with $\gamma(x) = \frac{p(x)}{q(x)}$, we obtain

$$\left\| I^{\alpha(x)} f(x) \right\|_{q(x)} \le c_4 \left\| Mf(x) \right\|_{p(x)}^{\Gamma}, \quad \Gamma = \sup_{x \in \Omega} \frac{p(x)}{q(x)} < 1 \tag{3.12}$$

(under the assumption that $\left\|(Mf(x))^{\frac{p(x)}{q(x)}}\right\|_{q(x)} \geq 1$, otherwise our theorem is already proved). In view of the assumed boundedness of the operator M in the space $L^{p((x)}(\Omega)$ the estimate (3.12) implies $\left\|I^{\alpha(x)}f(x)\right\|_{q(x)} \leq c_4\|M\|^{\Gamma}$. $\square$.

3.2. Theorems with prelimiting exponents.

Theorem 3.3. (Pre-limiting Sobolev theorem with a gap). *Let* $p(x) \in w - Lip(\bar{\Omega}), 1 \leq p_0 \leq p(x) \leq P < \infty$ *and* $\alpha(x) > 0$. *If*

$$\sup_{x\in\Omega} p(x)\left[\alpha(x) - \left(\frac{n}{p_0} - \frac{n}{P}\right)\right] \leq n \ , \tag{3.13}$$

the operator $I^{\alpha(x)}$ *is bounded from* $L^{p(x)}(\Omega)$ *into* $L^{r(x)}(\Omega)$ *where* $r(x)$ *is any function such that* $r(x) \geq 1$ *and*

$$\inf_{x\in\Omega}\left[\frac{1}{r(x)} - \frac{1}{p(x)} + \frac{\alpha(x)}{n}\right] > \frac{1}{p_0} - \frac{1}{P} \ . \tag{3.14}$$

Proof. We estimate the first term in the represenation (3.7) as follows:

$$|A_r(x)| \leq r^{\alpha(x)-\epsilon}\int_{\Omega}|f(y)||x-y|^{\epsilon-n}dy = r^{\alpha(x)-\epsilon}I^{\epsilon}(|f|) \tag{3.15}$$

and $\epsilon > 0$ is to be chosen in such a way that the operator I^{ϵ} to be bounded in the space $L^{p(x)}$. Theorem 2.8 gives a sufficient condition for that :

$$\epsilon > \frac{n}{p_0} - \frac{n}{P}. \tag{3.16}$$

Meanwhile the estimate (3.15) with the operator I^{ϵ} bounded in $L^{p(x)}$ means that we are exactly in the same situation as in the proof of Theorem 3.2 , the only difference being in the fact that the operator M must be replaced by I^{ϵ} and $\alpha(x)$ by $\alpha(x) - \epsilon$. Repeating the arguments of Theorem 3.2, we obtain that the operator $I^{\alpha(x)}$ is bounded frrom $L^{p(x)}$ to $L^{r(x)}$ where

$$\frac{1}{r(x)} = \frac{1}{p(x)} - \frac{\alpha(x)-\epsilon}{n} \tag{3.17}$$

and the condition

$$\sup_{x\in\Omega} p(x)\left[\alpha(x)-\epsilon\right] < n \tag{3.18}$$

is satisfied. (Naturally, we do not require that $p_0 > 1$, as in Theorem 3.2, since we do not deal with the maximal operator). It is easily seen that the set of conditions (3.17)-(3.18) is equivalent to the conditions (3.13)-(3.14). □

Remark 3.4. *In the case when $p(x)$ is constant, we have $P = p_0$, so that the "gap" $\frac{n}{p_0} - \frac{n}{P}$ is absent in (3.13)-(3.14) . Then the condition (3.13) admits validity of the equality $p(x)\alpha(x) = n$ on some set and then $r(x) = \infty$ on this set. In this relation we remark that it is of interest to investigate the spaces $L^{p(x)}$ which turn to be BMO at the set where $p(x) = \infty$.*

The following theorem, although dealing with the special case $r(x) = const$ is of independent interest, not being covered by Theorem 3.3.

Theorem 3.5. *Let $p(x) \in$ w-Lip $(\bar{\Omega})$, $1 < p_0 \leq p(x) \leq P < \infty$, and $0 < \alpha_0 \leq \alpha(x) \leq T < \infty$. If*

$$\sup_{x\in\Omega}\left(\frac{1}{p(x)} - \frac{\alpha(x)}{n}\right) < \inf_{x\in\Omega}\frac{1}{p(x)} \quad , \tag{3.19}$$

the operator $I^{\alpha(x)}$ is bounded from $L^{p(x)}(\Omega)$ into $L^r(\Omega)$ where the number $r \geq 1$ satisfies the inequality

$$\frac{1}{r} > \sup_{x\in\Omega}\left(\frac{1}{p(x)} - \frac{\alpha(x)}{n}\right) \quad . \tag{3.20}$$

Proof. We have

$$\left|I^{\alpha(x)}f(x)\right| \leq \int_\Omega \left(|f(y)|^{\frac{p(y)}{r}}|x-y|^{\epsilon(x)-\frac{n}{r}}\right)|f(y)|^{1-\frac{p(y)}{r}}|x-y|^{\epsilon(x)-\frac{n}{p(x)}}\,dy \tag{3.21}$$

where $p(x) = \frac{p(x)}{p(x)-1}$ and

$$\epsilon(x) = \frac{n}{2}\left[\frac{1}{r} + \frac{\alpha(x)}{n} - \frac{1}{p(x)}\right] > \frac{n\delta}{2} \tag{3.22}$$

with $\delta = \frac{1}{r} - \sup_{x\in\Omega}\left(\frac{1}{p(x)} - \frac{\alpha(x)}{n}\right) > 0$.

We apply the Holder inequality (1.21′) from [1] with the exponents $p_1(y) = r$, $p_2(y) = \frac{rp(y)}{r-p(y)}$, $p_3(y) = p'(y)$ and obtain

$$\left|I^{\alpha(x)}f(x)\right| \leq \left\|\,|f(y)|^{\frac{p(y)}{r}}|x-y|^{\epsilon(x)-\frac{n}{r}}\right\|_r \times N_1 \times N_2(x) \tag{3.23}$$

where
$$N_1 = \left\| |f|^{\frac{r-p(y)}{r}} \right\|_{\frac{rp(y)}{r-p(y)}}, \qquad N_2(x) = \left\| |x-y|^{\epsilon(x)-\frac{n}{p\prime(x)}} \right\|_{p(y)} .$$

Assuming that $\|f\|_p \le 1$, we obtain $N_1 \le \|f\|_p^{1-\frac{P}{r}} \le 1$ by means of (1.16) from [1] .

To estimate the norm $N_2(x)$, we apply the inequality (1.11) from [1] assuming that $N_2(x) \ge 1$. We obtain $N_2(x) \le \left\{ \int_\Omega |x-y|^{\beta(x)p\prime(y)} dy \right\}^{1-\frac{1}{P}}$ where $\beta(x) = \epsilon(x) - \frac{n}{p(x)}$. We denote $g(x,y) = |x-y|^{\beta(x)[p\prime(y)-p\prime(x)]}$. Here $p\prime(x) \in$ w-Lip $(\bar{\Omega})$, since $\inf p(x) > 1$. Therefore, $|\log g(x,y)| \le A|\beta(x)|$ where $|x-y| < 1$ and A is the constant from the w-Lip - condition for the function $p(x)$. Then $0 < c_1 \le g(x,y) \le c_2 < \infty$ for $|x-y| < 1$ and, consequently, for all $x, y \in R^n$. So,

$$N_2(x) \le c_3 \left\{ \int_\Omega |x-y|^{\beta(x)p\prime(x)} dy \right\}^{1-\frac{1}{P}}$$

Since $\beta(x)p\prime(x) \ge \frac{n\delta}{2} - n$ in view of (3.22), we easily obtain that $N_2(x) \le c_4 = const$. Hence, the inequality (3.23) yields

$$\left\| I^{\alpha(x)} f \right\|_r \le c_4 \left\{ \int_\Omega |f(y)|^{p(y)} dy \int_\Omega |x-y|^{r\epsilon)y)-n} dx \right\}^{\frac{1}{r}} .$$

The inner integral is bounded, since $r\epsilon(y) > \frac{rn\delta}{2} > 0$. Consequently, $\left\| I^{\alpha(x)} f \right\|_r \le c_5, \quad \|f\|_p \le 1$. □

3.3. Necessary conditions for the operator $I^{\alpha(x)}$ to be bounded from $L^{p(x)}$ into $L^{q(x)}$.

Theorem 3.6. *Let $1 \le p_0 \le p(x) \le P < \infty$ and $0 < \alpha_0 \le \alpha(x) \le T < \infty, \quad x \in R^n$. If the operator $I^{\alpha(x)}$ over $\Omega = R^n$ is bounded from $L^{p(x)}(R^n)$ into $L^{q(x)}(R^n)$ with $1 \le q(x) \le Q < \infty, \ x \in R^n$, for some $Q > 1$, then*

$$\inf_{x \in R^n} \left[\alpha(x) + \frac{n}{q(x)} \right] \le \frac{n}{p_0} , \tag{3.24}$$

$$\sup_{x \in R^n} \left[\alpha(x) + \frac{n}{q(x)} \right] \ge \frac{n}{P} . \tag{3.25}$$

In case of the spaces $L^{p(x)}(\Omega)$ on a bounded domain Ω only the condition (3.25) (with the supremum over Ω) is necessary.

The proof of the theorem uses the dilatation operator $\Pi_t f(x) = f(tx), x \in R^n, t > 0$ (although it does not preserve the space $L^{p(x)}$), the estimates

$$t^{-\frac{n}{u}}\|f\|_{p_t} \leq \|\Pi_t f\|_p \leq t^{-\frac{n}{v}}\|f\|_{p_t}$$

where $p_t = p(\frac{x}{t})$ and $u = P, v = p_0$ for $t < 1$ and $u = p_0, v = P$ for $t > 1$, the relation $I^{\alpha(x)}\Pi_t f = t^{-\alpha(x)} I^{\alpha_t(x)} f$, $\alpha_t(x) = \alpha(\frac{x}{t})$, and the estimates

$$t^{-m_p}\|f\|_{p_t} \leq \| \ t^{-\alpha(x)}\Pi_t f\|_p \leq t^{-M_p}\|f\|_{p_t}, \quad 0 < t < 1, \tag{3.26}$$

$$t^{-M_p}\|f\|_{p_t} \leq \| \ t^{-\alpha(x)} \ Pi_t f\|_p \leq t^{-m_p}\|f\|_{p_t}, \quad t > 1, \tag{3.27}$$

with

$$m_p = \inf_{x \in R^n}\left[\alpha(x) + \frac{n}{p(x)}\right], \quad M_p = \sup_{x \in R^n}\left[\alpha(x) + \frac{n}{p(x)}\right]$$

which can be obtained straightforwardly.

Remark 3.7. *It seems to be natural to suppose that the conditions (3.24)-(3.25) which do not reach the case of the limiting Sobolev exponent, can be strengthened. The question, however, remains open. These conditions are satisfied, for example, by any perturbed Sobolev exponent*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha(x)}{n} + h(x) \tag{3.28}$$

with the perturbation $|h(x|) \leq \frac{1}{p_0} - \frac{1}{P}$. This can be easily seen from the fact that the conditions (3.24)-(3.25) for the function (3.28) take the form

$$\inf_{x \in R^n}\left[\frac{1}{p(x)} + h(x)\right] \leq \sup_{x \in R^n}\frac{1}{p(x)} \ , \quad \sup_{x \in R^n}\left[\frac{1}{p(x)} + h(x)\right] \geq \inf_{x \in R^n}\frac{1}{p(x)}$$

and the condition $|h(x)| \leq \frac{1}{p_0} - \frac{1}{P}$ is sufficient for validity of the above inequalities.

3.4. A counterexample to the Sobolev theorem in case of discontinuous exponent $p(x)$.

The Sobolev theorem in the form $I^{\alpha(x)}\ :\ L^{p(x)}\to L^{q(x)},\ \ \frac{1}{q(x)}=\frac{1}{p(x)}-\frac{\alpha(x)}{n}$, is not, in general, true in case of discontinuous exponents $p(x)$ (even if $\alpha(x)=$ *const*), see Lemma 3.8 below. Probably, continuity of $p(x)$ is a necessary condition . In any case, the following lemma shows that $p(x)$ cannot be piece-wise constant. Let $n=1$ for simplicity, $\Omega=[-1,1]$ and let

$$p(x)=p_1,\ \alpha(x)=\alpha_1,\ x<0;\ \ p(x)=p_2,\ \alpha(x)=\alpha_2,\ x>0. \tag{3.29}$$

Lemma 3.8. *If the operator*

$$I^{\alpha(x)}f(x)=\int_{-1}^{1}\frac{f(y)dy}{|x-y|^{1-\alpha(x)}} \tag{3.30}$$

is bounded from $L^{p(x)}[-1,1]$ *into* $L^{q(x)}[-1,1]$, $\frac{1}{q(x)}=\frac{1}{p(x)}-\alpha(x),\ p_i\alpha_i<1, i=1,2,$ *then necessarily* $p_1=p_2$.

Proof. In view of Lemma 1.14 from [1], boundedness of the operator (3.30) stated in the lemma, is equivalent to boundedness of the folowing 4 operators in the corresponding setting:

$$A_{kj}\varphi=\int_0^1\frac{\varphi(y)dy}{|x-(-1)^{k+j}y|^{1-\alpha_k}}\ \ :\ L^{p_j}[0,1]\to L^{q_k}[0,1],\quad k,j=1,2.$$

Meanwhile, it is well known that for the operator A_{kj} to be bounded from $L^{p_j}[0,1]$ into $L^{q_k}[0,1]$, it is necessary that $\frac{1}{q_k}\ge\frac{1}{p_j}-\alpha_k;\quad k,j=1,2;$ where the cases $k=1,j=2$ and $k=2,j=1$ yield the condition $p_1=p_2$.

Remark 3.9. *Simple modification of Lemma 3.8 shows that, in case of a piece-wise constant exponent* $p(x)$*, even the "prelimiting" theorem cannot hold with an arbitrarily small "gap"*

$$\epsilon(x)=\frac{1}{r(x)}-\frac{1}{q(x)}$$

which is $\epsilon_1=\frac{1}{r_1}-\frac{1}{q_1},\ x<0,\ \epsilon_2=\frac{1}{r_2}-\frac{1}{q_2},\ x>0.$ *Really, in the same way as in the proof of Lemma 3.8 it can be shown that necessarily either* $\epsilon_1\ge\frac{1}{p_2}-\frac{1}{p_1}$ *or* $\epsilon_2\ge\frac{1}{p_1}-\frac{1}{p_2}$.

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