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Convolution type operators in $L^{p(x)}$.

Introduction

We intend to investigate integral transforms of convolution type or operators close to them within the frameworks of the spaces $L^{p(x)}$ of variable order $p(x)$. We note at once that convolution operators have a "bad" behavior in these spaces. The reason is in the fact that the convolution operator $K = k * f$, in general, shifts a singularity a function f has at some point, to another point, while the integrability exponent $p(x)$ may have different values at those points. Naturally, it depends on the kernel $k(x)$ whether it shifts singularities or not. As a result, Young type theorem K : $L^{p(x)}(R^n) \to L^{p(x)}(R^n)$ for an arbitrary summable kernel $k(x)$ is not already valid for the spaces $L^{p(x)}(R^n)$.

This paper may be considered as a preliminary one. It provides a necessary tool for our future investigations of convolution and potential type operators. In the next paper "Convolution and potential type operators in $L^{p(x)}$ " we shall prove some kind of Young Theorem for these spaces , consider also potential type operators and treat the question of validity of Sobolev-type theorem in the spaces $L^{p(x)}(R^n)$. It should be emphasized that we shall deal with the potential type operators whose order $\alpha(x)$ is variable as well.

The approach to the investigation of potential type operators is based on estimates of $L^{p(x)}$ - norms of power functions of distance truncated to the exterior of the ball, in dependence of its radius. We develop these estimates in this paper (section 2) and shall continue them in the next paper together with applications to boundedness problems of the convolution and potential type operators within the farmework of the spaces $L^{p(x)}$.

In this paper we develop some results for the spaces $L^{p(x)}$ themselves both for our needs in application and for completeness of the presentation (section 1). The spaces $L^{p(x)}$ were first investigated in [8]. Some further developments of the theory of $L^{p(x)}$ - spaces were given in [5], [7].

N o t a t i o n . Ω is a measurable set in R^n, $|\Omega|$ is its Lebesgue measure, $\dot{R}^n$ is the compactification of R^n by the unique infinite point; $S_{n-1} = \{x \in R^n : |x| = 1\}$; $B(x_0, r)$ is the ball cenetered at x_0 and of radius r ; $B_n = B(0,1)$; $p(x_0, r) = \inf_{|y-x_0|\leq r} p(y), P(x_0, r) = \sup_{|y-x_0|\leq r} p(y)$, $\bar{p}(x_0, r) = \inf_{|y-x_0|\geq r} p(y), \bar{P}(x_0, r) = \sup_{|y-x_0|\geq r} p(y); \chi_\Omega(x) \equiv 1$ for $x \in \Omega$ and $\equiv 0$ for $x \notin \Omega$; $f_\beta(x) = f_\beta(x_0, r; x) =| x - x_0 |^{\beta(x_0)} \chi_{B(x_0,r)}(x), g_\beta(x) = g_\beta(x_0, r; x) =| x - x_0 |^{\beta(x_0)} [1 - \chi_{B(x_0,r)}(x)]; \lambda_\beta = \|f_\beta\|, \mu_\beta = \|g_\beta\|$.

I. Basics of the theory of the spaces $L^{p(x)}(\Omega)$

1.1. Metric in $L^{p(x)}(\Omega)$

Let Ω be a measurable set in R^n and $p(x)$ be a non-negative measurable function on Ω. Let $E_a = E_a(p) := \{x \in \Omega : p(x) = a\}$ where we shall be interested in the cases $a = 0$, $a = 1$ and $a = \infty$. Everywhere below it is assumed that $| E_0 | = 0$.

Definition 1.1. *By $L^{p(x)}(\Omega)$ we denote the set of measurable functions $f(x)$ on Ω such that*

$$I_p(f) := \int_{\Omega\setminus E_\infty} | f(x) |^{p(x)} dx < \infty \tag{1.1}$$

and $\sup_{x\in E_\infty} | f(x) | < \infty$.

Following [8] we consider a natural topology in $L^{p(x)}$ defined by the convergence

$$\int_{\Omega\setminus E_\infty} | f_m(x) - f(x) |^{p(x)} dx + \sup_{x\in E_\infty} | f_m(x) - f(x) | < \epsilon. \tag{1.2}$$

We shall essentially use the numbers

$$P = \sup_{x\in\Omega\setminus E_\infty} p(x), \quad p_0 = \inf_{x\in\Omega} p(x) \tag{1.3}$$

so that $0 \leq p_0 \leq P \leq \infty$. In case $P < \infty$ we set

$$d(f, g) = \{\int_{\Omega\setminus E_\infty} |f(x) - g(x)|^{p(x)}\}^{1/P_1} + \sup_{x\in E_\infty} |f(x) - g(x)|$$

with $P_1 = max\{P, 1\}$. We shall also use the notation $\rho(f) = \rho_p(f) = d(f, 0)$.

Lemma 1.2. [8] . *The topological space defined by Definition 1.1 and (1.2), is linear if and only if* $P < \infty$ *and then* $d(f, g)$ *is a metric on this space.*

Let $S = S(\Omega)$ be the set of simple step functions $\sum_{k=1}^{N} c_k \chi_{\Omega_k}(x)$ where Ω_k are arbitrary measurable bounded sets in Ω and $\chi_{\Omega_k}(x)$ are their characteristic functions. Evidently, $S \subset L^{p(x)}$ if $P < \infty$.

1.2. Kolmogorov-Minkowski-type norm in $L^{p(x)}(\Omega)$, $1 \le p(x) \le \infty$.

Theorem 1.4 below introduces a norm inspired by the Kolmogorov's theorem on norming topological spaces ([4];[3], Ch. 4 , p. 122) with a convex bounded neighbourhood of the null-element, the Minkowsky functional of this neighbourhood being a norm.

Lemma 1.3 ([8],[7]). *Let* $f(x) \in L^{p(x)}(\Omega)$, $0 \le p(x) \le \infty$. *The function*

$$F(\alpha) \ := \ I_p\left(\frac{f}{\lambda}\right) \ , \ \lambda > 0 \ , \tag{1.4}$$

takes finite values for all $\lambda \ge 1$, *is continuous and decreases and* $\lim_{\lambda\to\infty} F(\lambda) = 0$. *If* $P < \infty$, *the same is true for all* $\lambda > 0$.

Theorem 1.4 ([8],[7]). *Let* $0 \le p(x) \le \infty$. *For any* $f(x) \in L^{p(x)}(\Omega)$ *the functional*

$$\|f\|_{(p)} \ = \ inf\left\{\lambda : \lambda > 0, \ \ \int_{\Omega\setminus E_\infty}\left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\} \tag{1.5}$$

takes a finite value and

$$I_p\left(\frac{f}{\|f\|_{(p)}}\right) \ \le \ 1 \ , \ \ \|f\|_{(p)} \ne 0 \quad . \tag{1.6}$$

If either $P < \infty$ *or* $\|f\|_{(p)} \ge 1$, *then*

$$I_p\left(\frac{f}{\|f\|_{(p)}}\right) \ = \ 1 \quad , \ \|f\|_{(p)} \ne 0 \ \ . \tag{1.7}$$

Finally, if $1 \le p(x) \le P < \infty$, $x \in \Omega\setminus E_\infty$, *then*

$$\|f\|_p \ = \ \|f\|_{(p)} \ + \ \sup_{x\in E_\infty} |\, f(x)| \tag{1.8}$$

is a norm in $L^{p(x)}(\Omega)$.

Corollary 1. *The functional (1.5) satisfies the estimates*

$$\left(\frac{\|f\|_{(p)}}{\lambda}\right)^P \le I_p\left(\frac{f}{\lambda}\right) \le \left(\frac{\|f\|_{(p)}}{\lambda}\right)^{p_0} \ , \ \lambda \ge \ \|f\|_{(p)} \ , \tag{1.9}$$

$$\left(\frac{\|f\|_{(p)}}{\lambda}\right)^{p_0} \le I_p\left(\frac{f}{\lambda}\right) \le \left(\frac{\|f\|_{(p)}}{\lambda}\right)^P \ , \ 0 < \lambda \le \ \|f\|_{(p)} \ , \tag{1.10}$$

where the cases $p_0 = 0$ *or* $P = \infty$ *are admitted.*

Proof. Rewriting (1.9) and (1.10) as $\lambda^P \le I_p\left(\frac{\lambda}{\|f\|}f\right) \le \lambda^{p_0}$, $\lambda \le 1$, and $\lambda^{p_0} \le I_p\left(\frac{\lambda}{\|f\|}f\right) \le \lambda^P$, $\lambda \ge 1$, we see that the last inequalities follow from (1.7) if either $P < \infty$ or $P = \infty$ but $\|f\|_{(p)} \ge 1$. If $P = \infty$ and $\|f\|_{(p)} \le 1$, we should refer to (1.6) and take into account that $\|g\|_{(p)} = \lambda \ge 1$ for $g(x) = \frac{\lambda f(x)}{\|f\|_{(p)}}$.

Corollary 2. *For any* $p(x)$, $0 \le p_0 \le p(x) \le P < \infty$, $x \in \Omega \backslash E_\infty$, *the following estimates hold*

$$\|f\|_{(p)}^P \le I_p(f) \le \|f\|_{(p)}^{p_0} \ , \ \|f\|_{(p)} \ \le \ 1, \tag{1.11}$$

$$\|f\|_{(p)}^{p_0} \le I_p(f) \le \|f\|_{(p)}^P \ , \ \|f\|_{(p)} \ \ge \ 1, \tag{1.12}$$

Corollary 3. *Let* E *be a measurable set in* $\Omega \backslash E_\infty$ *and let* $\chi_E(x)$ *be its characteristic function. If* $0 < p_0 \le P < \infty$, *we have*

$$|E|^{1/p_0} \ \le \ \|\chi_E\|_{(p)} \ \le \ |E|^{1/P} \ , \ |E| \ge 1,$$

signs of the inequalities being opposite if $|E| \le 1$, *so that the equality* $\|\chi_E\|_{(p)} = 1$ *is equivalent to* $|E| = 1$. *(If, instead of* $E \subseteq \Omega \backslash E_\infty$, *it is assumed that* $E \subseteq \Omega$, *then* $|E|$ *must be replaced by* $|E \bigcap (\Omega \backslash E_\infty)|$.

Remark 1.5. *An example that illustrates just (1.6) instead of (1.7) is* $\Omega = [0,1]$, $p(x) = \frac{1}{x}$, $f(x) = 4^{-x}x^{-x/2}$.

Remark 1.6 ([8]). *In case* $P = \infty$, *the functional* $\|f\|_{(p)}$ *does exist for any* $f \in L^{p(x)}$ *according to Theorem 1.1 . However, if* $\|f\|_{(p)} < \infty$, *it does not necessarily implies that* $f \in L^{p(x)}(\Omega \backslash E_\infty)$, *but* $f(x) \in \mathcal{L}L^{p(x)}(\Omega \backslash E_\infty)$ *where* $\mathcal{L}L^{p(x)}$ *denotes the linear envelope of the class* $L^{p(x)}(\Omega \backslash E_\infty)$. .

Remark 1.7. *A realization of Kolmogorov-Minkowsky norm for the Orlicz spaces, similar to (1.8), is known in the theory of Orlicz spaces [6] as the Luxemburg norm.*

Remark 1.8. *The space $L^{p(x)}(\Omega)$ is ideal, which means that it is complete and the inequality $|f(x)| \leq |g(x)|$ implies $\|f\|_p \leq \|g\|_p$* (see the proof of completeness of the space $L^{p(x)}(\Omega)$ in [5],[7]).

Remark 1.9 ([8]). *The (semi)norm $\|f\|_{(p)}$, $p(x) \geq 1$, is strict in the sense that the equality $\|f+g\|_{(p)} = \|f\|_{(p)} + \|g\|_{(p)}$ is possible if and only if $g = cf, c > 0$ $(c = \|g\|_{(p)}/\|f\|_{(p)})$.*

Remark 1.10. *Let $1 \leq p(x) \leq \infty$, $P < \infty$. The (semi)norm $\|f\|_{(p)}$ may be represented in the form*

$$\|f\|_{(p)} = \int_{\Omega\setminus E_\infty} \varphi_0(x) f(x) dx \ , \ \varphi_0(x) \in L^{q(x)}(\Omega) \tag{1.13}$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$, $\varphi_0(x) = \left|\frac{f(x)}{\|f\|_{(p)}}\right|^{p(x)-1} \frac{f(x)}{|f(x)|}$, $x \notin E_\infty$, and $\|\varphi_0\| \leq 1$.

Lemma 1.11 ([7]). *Let $0 < p_0 \leq P \leq \infty$. If*

$$I_p\left(\frac{f}{a}\right) \ < \ b \ , \quad a > 0, \ b > 0, \tag{1.14}$$

then $\|f\|_{(p)} \leq ab^\nu$ with $\nu = 1/p_0$ if $b \geq 1$ and $\nu = 1/P$ if $b \leq 1$.

Lemma 1.12 ([7]). *Let $0 < \gamma(x) \leq p(x) \leq P < \infty$, $x \in \Omega\setminus E_\infty$. Then*

$$\|f\|_{(p)}^{\gamma_0} \leq \|f^\gamma\|_{(\frac{p}{\gamma})} \leq \|f\|_{(p)}^{\Gamma} \ , \ \|f\|_{(p)} \geq 1 \ , \tag{1.15}$$

$$\|f\|_{(p)}^{\Gamma} \leq \|f^\gamma\|_{(\frac{p}{\gamma})} \leq \|f\|_{(p)}^{\gamma_0} \ , \ \|f\|_{(p)} \leq 1 \ , \tag{1.16}$$

where $f^\gamma = |f(x)|^{\gamma(x)}$ and $\gamma_0 = \inf_{x\in\Omega\setminus E_\infty} \gamma(x)$, $\Gamma = \sup_{x\in\Omega\setminus E_\infty} \gamma(x)$. If $p(x)$ and $\gamma(x)$ are continuous on $\Omega\setminus E_\infty$, there exists a point $x_0 \in \Omega\setminus E_\infty$ such that $\|f^\gamma\|_{(\frac{p}{\gamma})} = \|f\|_{(p)}^{\gamma(x_0)}$.

Corollary. *Let $0 \leq p_0 \leq p(x) \leq P < \infty$, $x \in \Omega\setminus E_\infty$. If $p(x)$ is continuous on $\Omega\setminus E_\infty$, there exists a point $x_0 \in \Omega\setminus E_\infty$ (depending on f) such that*

$$\|f\|_{(p)} = \left\{\int_{\Omega\setminus E_\infty} |f(x)|^{p(x)} dx\right\}^{\frac{1}{p(x_0)}} . \tag{1.17}$$

Lemma 1.13. *Let $0 < p_1(x) \leq p(x) \leq p_2(x) \leq \infty$ and $|E_\infty(p_2)| = 0$. Then*

$$L^{p_1(x)}(\Omega) \bigcap L^{p_2(x)}(\Omega) \subseteq L^{p(x)}(\Omega) \subseteq L^{p_1(x)}(\Omega) \ + \ L^{p_2(x)}(\Omega).$$

where the algebraic sum of spaces stands in the right-hand side. Besides, $\|f\|_p \le \max\{\|f\|_{p_1}, \|f\|_{p_2}\}$.

Proof is straightforward.

The property of semiadditivity of the norm:

$$\max\{\|f\|_{L^{p(x)}(\Omega_1)}, \|f\|_{L^{p(x)}(\Omega_2)}\} \le \|f\|_{L^{p(x)}(\Omega)} \le \|f\|_{L^{p(x)}(\Omega_1)} \ + \ \|f\|_{L^{p(x)}(\Omega_2)} \tag{1.18}$$

with $\Omega_1 \bigcup \Omega_2 = \Omega$, well known for the case of constant exponents, is covered by the following lemma.

Lemma 1.14. *Let* $\Omega = \Omega_1 \bigcup \Omega_2$ *and let* $p(x)$ *be a function on* Ω, $p(x) \ge 1$ *and* $P < \infty$. *Then (1.18) holds for any* $f(x) \in L^{p(x)}(\Omega)$.

Proof. Let $|E_\infty| = 0$ for simplicity. We denote $a = \|f\|_{L^{p(x)}(\Omega_1)}, b = \|f\|_{L^{p(x)}(\Omega_2)}$. Let $a \ge b$ for definiteness. We have

$$\int_\Omega \left|\frac{f(x)}{\max(a,b)}\right|^{p(x)} dx \ge \int_{\Omega_1} \left|\frac{f(x)}{a}\right|^{p(x)} = 1.$$

Hence $\|f\|_{L^{p(x)}(\Omega)} \ge \max(a,b)$. To prove the right-hand side inequality, we put

$$\frac{f(x)}{a+b} = \frac{a}{a+b}\frac{\chi_1(x)f(x)}{a} + \frac{b}{a+b}\frac{\chi_2(x)f(x)}{b}$$

where $\chi_i(x)$ are the charecteristic functions of the sets $\Omega_i, i = 1,2$. Using the convexity property, we obtain $\int_\Omega \left|\frac{f(x)}{a+b}\right|^{p(x)} dx \le 1$ which was required.

In case $|E_\infty| > 0$, the arguments are similar if we take into account the fact that the lemma has already been proved for the situation $\Omega \backslash E_\infty = \Omega_1^* \bigcup \Omega_2^*$ where $\Omega_i^* = \Omega_i \backslash E_\infty \bigcap \Omega_i, i = 1,2$. □

1.3. Another version of the Kolmogorov-Minkowskii norm.

The Kolmogorov-Minkowski-type norm can be also introduced directly with respect to the whole set Ω :

$$\|f\|_p^1 \ = \ \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \ + \ \sup_{x \in E_\infty}\left|\frac{f(x)}{\lambda}\right| \ \le \ 1\right\} \tag{1.19}$$

which is well defined for $f(x) \in L^{p(x)}(\Omega)$, whatever measurable function $p(x), 0 \le p(x) \le \infty$, is used ; it is a norm, if $1 \le p(x) \le \infty$. This can be proved similarly to Theorem 1.4, see [7]. Analogously to (1.7) it can be proved that

$$\int_{\Omega\setminus E_\infty} \left|\frac{f(x)}{\|f\|_p^1}\right|^{p(x)} dx \quad + \quad \frac{\|f\|_{L^\infty(E_\infty)}}{\|f\|_p^1} \ = \ 1 \tag{1.20}$$

if $P<\infty$ or $P=\infty$, but $\|f\|_p^1 \geq 1$. It is exactly this version of the norm that was used in [5].

Theorem 1.15 ([7]). *The norms (1.8) and (1.20) are equivalent:* $\frac{1}{2}\,\|f\|_p \leq \|f\|_p^1 \leq \|f\|_p$ *where* $f(x)\in L^{p(x)}(\Omega),\ 1\leq p(x)\leq\infty,\ P<\infty.$

1.4. Holder inequality and its generalizations.

Theorem 1.16 ([8],[7]). *Let* $f(x)\in L^{p(x)}(\Omega), 1\leq p(x)\leq\infty,$ *and* $\varphi(x)\in L^{q(x)}(\Omega), \frac{1}{p(x)}+\frac{1}{q(x)}\equiv 1, x\in\Omega.$ *Then*

$$\int_\Omega |f(x)\varphi(x)|dx \leq k\|f\|_p\ \|\varphi\|_q \tag{1.21}$$

with $k=\frac{1}{p_0}+\frac{1}{q_0}=\sup\frac{1}{p(x)}+\sup\frac{1}{q(x)}$. *The Holder inequality holds also in the form*

$$\int_\Omega |f_1(x)...f_m(x)|dx \leq c\|f_1\|_{p^1}...\|f_m\|_{p^m} \tag{1.21′}$$

where $p^1(x)\geq 1,...,p^m(x)\geq 1$ *and* $\sum_{k=1}^m 1/p^k(x)\equiv 1,\ x\in\Omega,$ *and* $c=\sum_{k=1}^m 1/p_0^k, p_0^k=\min_{x\in\Omega} p^k(x).$

Remark 1.17. *If instead of (1.5) we introduce the (semi)norm* $\|f\|_{(p)}$ *as*

$$\|f\|_{(p)} \ = \ inf\left\{\lambda>0: \ \int_{\Omega\setminus E_\infty}\frac{2}{p(x)}\left|\frac{f(x)}{\lambda}\right|^{p(x)} dx\leq 1\right\} \tag{1.22}$$

then the Holder inequality (1.21) holds with the constant 1 :

$$\int_\Omega |f(x)\varphi(x)|dx \leq \|f\|_p\ \|\varphi\|_q \tag{1.23}$$

In case of constant $p(x)=p$ the Holder inequality has a simple generalization in the form $\|uv\|_r\leq\|u\|_p\|v\|_q, \frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, which is an immediate consequence of the Holder inequality itself. In case of variable exponents such an inequality does not follow from the Holder inequality since $\||u|^r\| \neq \|u\|_p^r$, see Lemma 1.12. However, it is valid , but may be obtained not as a consequence of the Holder inequality.

Lemma 1.18 ([7]). *Let* $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv \frac{1}{r(x)}, \ p(x) \geq 1, \ q(x) \geq 1, \ r(x) \geq 1$ *and let* $R = \sup_{x\in\Omega\setminus E_\infty(r)} r(x) < \infty$. *Then*

$$\|uv\|_r \leq c\|u\|_p\|v\|_q \tag{1.24}$$

for all $u \in L^{p(x)}$ *and* $v \in L^{q(x)}$ *with* $c = c_1 + c_2, c_1 = \sup_{x\in\Omega\setminus E_\infty(r)} \frac{r(x)}{p(x)}, \ c_2 = \sup_{x\in\Omega\setminus E_\infty(r)} \frac{r(x)}{q(x)}$.

Remark 1.19. *If we would use the modification (1.22) of the (semi)norm* $\|f\|_{(p)}$, *then the inequality (1.24) holds with* $c = 2$ *and the requirement* $R < \infty$ *may be omitted .*

1.5. On the imbedding $L^{p(x)} \subseteq L^{r(x)}$.

Theorem 1.20 ([7]). *Let* $0 \leq r(x) \leq p(x) \leq \infty$ *and let* $|\Omega\setminus E_\infty(r)| < \infty$. *If* $E_\infty(r) \subseteq E_\infty(p)$ *and* $R := \sup_{x\in E_\infty(p)\setminus E_\infty(r)} r(x) < \infty$, *then* $L^{p(x)}(\Omega) \subseteq L^{r(x)}(\Omega)$ *and*

$$I_r(f) \leq I_p(f) + |E_\infty(p)\setminus E_\infty(r)|\|f\|^R_{L^\infty(E_\infty(p)\setminus E_\infty(r))} \ + \ |\Omega\setminus E_\infty(r)| \tag{1.25}$$

for any $f \in L^{p(x)}$. *(In the case* $E_\infty(p) = E_\infty(r)$, *the second term in the right hand side should be omitted and* R *is allowed to be infinite). If, moreover,* $1 \leq r(x) \leq p(x)$ *and* $E_\infty(p) = E_\infty(r)$, *the inequality for norms also holds:*

$$\|f\|_{(r)} \ \leq \ c_0^\nu\|f\|_{(p)} \tag{1.26}$$

where $c_0 = c_2+(1-c_1)|\Omega\setminus E_\infty(p)|, c_1 = \inf_{x\in\Omega\setminus E_\infty(p)} \frac{r(x)}{p(x)}, c_2 = \sup_{x\in\Omega\setminus E_\infty(p)} \frac{r(x)}{p(x)}$, *and* $\nu = \frac{1}{r_0}$ *if* $c_0 \geq 1$ *and* $\nu = \frac{1}{R}$ *if* $c_0 \leq 1$.

We note that in [5] it was shown, under the assumption $|\ \Omega\ | < \infty$, that continuous imbedding holds if and only if $r(x) \leq p(x)$.

1.6. Riesz-type norm in $L^{p(x)}(\Omega)$.

We consider now the norm inspired by the Riesz theorem on the representation of a linear functional in L^p. We introduce first the space

$$\tilde{L}^{p(x)}(\Omega) \ = \ \left\{f(x) : \left|\int_\Omega f(x)\varphi(x)dx\right| < \infty \ \ \forall\varphi(x) \in L^{q(x)}(\Omega)\right\} \tag{1.27}$$

where $1 \leq p(x) \leq \infty$ and $\frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1$. This space will in fact coincide with $L^{p(x)}(\Omega)$ under some natural assumptions on $p(x)$ as it reaches the values 1 and ∞ . The imbedding $L^{p(x)} \subseteq \tilde{L}^{p(x)}(\Omega)$, $1 \leq p(x) \leq \infty$, is an immediate consequence of the Holder inequality (1.21).

We note that the space (1.27) is always linear. So , by Lemma 1.2 , it cannot coincide with $L^{p(x)}$ *á priori* if $P = \infty$.

Besides the notations p_0 and P for $p(x)$, see (1.4), and q_0 and Q for $q(x)$, we shall also use

$$p_0^1 \;=\; \inf_{x \in \Omega \setminus E_1(p)} p(x) \;, \qquad q_0^1 \;=\; \inf_{x \in \Omega \setminus E_1(q)} q(x) \;.$$

Evidently,

$$E_1(p) = E_\infty(q),\; E_1(q) = E_\infty(p),\; Q = \frac{p_0^1}{p_0^1 - 1},\; q_0^1 = \frac{P}{P-1} \;. \tag{1.28}$$

The space (1.27) can be equipped with the natural norms

$$\|f\|_p^* \;=\; \sup_{\rho_q(\varphi) \leq 1} \left| \int_\Omega f(x)\varphi(x)dx \right|, \tag{1.29}$$

$$\|f\|_p^{**} \;=\; \sup_{\|\varphi\|_q \leq 1} \left| \int_\Omega f(x)\varphi(x)dx \right|, \tag{1.30}$$

where the distance $\rho_q(\varphi)$, defined in Subsection 1.1 is taken with respect to the variable exponent $q(x)$ and it is assumed that $Q < \infty$ (that is $p_0^1 > 1$) in (1.29), while $p(x)$ may be arbitrary ($1 \leq p(x) \leq \infty$) in case of (1.30). The first of these norms was used in [5] , but with $\|\varphi\|_q^1$ instead of $\|\varphi\|_q$ (see (1.19)).

Note that from Remark 1.10 it follows that

$$\|f\|_p \leq \|f\|_p^{**} \tag{1.31}$$

in case $1 \leq p(x) \leq P < \infty$, $|E_\infty| = 0$.

Lemma 1.21 ([7]). *Let $f(x) \in \tilde{L}^{p(x)}(\Omega), p_1^0 > 1$. Then $\|f\|_p^* < \infty$ and*

$$\int_\Omega |f(x)\varphi(x)|dx \leq \|f\|_p^* \|\varphi\|_q^1 \leq \|f\|_p^* \|\varphi\|_q \tag{1.32}$$

for all $\varphi(x) \in L^{q(x)}(\Omega), 1/p(x) + 1/q(x) \equiv 1$, where $\|\varphi\|_q^1$ is the norm (1.27). Besides, the functional (1.29) is a norm in $\tilde{L}^{p(x)}(\Omega)$.

Lemma 1.22 ([7]). *Let* $1 \le p(x) \le \infty,\ p_0^1 > 1$ *and* $P < \infty$. *The norms (1.29) and (1.30) are equivalent on functions* $f(x) \in \tilde{L}^{p(x)}(\Omega)$ *:*

$$2^{1-Q/q_0^1}\|f\|_p^{**} \ \le\ \|f\|_p^{*} \ \le\ \|f\|_p^{**}\ . \tag{1.33}$$

They coincide with each other in the cases : 1) $|E_1(p)| = 0$*, 2)* $p(x) = const$ *for* $x \in \Omega\backslash(E_\infty \bigcup E_1)$.

Theorem 1.23 ([7]). *Let* $p_0^1 > 1$ *and* $P < \infty$. *The spaces* $L^{p(x)}(\Omega)$ *and* $\tilde{L}^{p(x)}(\Omega)$ *coincide up to the equivalence of norms:*

$$\frac{1}{3}\ \|f\|_p \ \le\ \|f\|_p^{*} \ \le\ \left(\frac{1}{p_0} + \frac{1}{q_0}\right)\|f\|_p \tag{1.34}$$

where 1/3 may be replaced by 1 if $|E_1| = |E_\infty| = 0$.

Corollary. *Let* $f(x) \in L^{p(x)}(\Omega),\ \varphi(x) \in L^{q(x)}(\Omega),\ 1 \le p(x) \le \infty,\ \frac{1}{p(x)} + \frac{1}{q(x)} \equiv 1, x \in \Omega$. *With respect to the norms (1.29)-(1.30) the Holder inequality holds with the multiple constant 1 :*

$$\int_\Omega |f(x)\varphi(x)|dx \le \|f\|_p^{*}\|\varphi\|_q\ ,\ p_0^1 > 1, \tag{1.35}$$

$$\int_\Omega |f(x)\varphi(x)|dx \le \|f\|_p^{**}\|\varphi\|_q. \tag{1.36}$$

The inequality

$$\int_\Omega |f(x)\varphi(x)|dx \le \|f\|_p^{*}\|\varphi\|_q^{*} \tag{1.37}$$

is also valid in case

$$p_0^1 > 1,\ P < \infty,\ |E_\infty(p)| = |E_1(p)| = 0. \tag{1.38}$$

Really, (1.35) has already been given in (1.32) while (1.36) follows directly from the definition (1.30). The inequality (1.37) follows from (1.35) since $\|\varphi\|_q \le \|\varphi\|_q^{*}$ under the conditions (1.38) in view of Theorem 1.23.

The definition (1.27) gives one of posiible ways to define the space $L^{p(x)}(\Omega)$ as linear in case $P = \infty$. It could be also defined from the very beginning as a linear envelope of the space $L^{p(x)}(\Omega)$ (see Remark 1.6) or as

$$\hat{L}^{p(x)}(\Omega) = \left\{ f(x) : \exists\lambda > 0 \Longrightarrow \int_{\Omega\backslash E_\infty} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx\ +\ \|f\|_{L^\infty(E_\infty)} < \infty \right\} \tag{1.39}$$

This space is always linear, $0 \le p(x) \le \infty$. So, in case $P = \infty$ we can deal with three versions of definitions - $\mathcal{L}L^{p(x)}, \hat{L}^{p(x)}$ and $\tilde{L}^{p(x)}$. It is easily seen that $\mathcal{L}L^{p(x)} = \hat{L}^{p(x)} \subseteq \tilde{L}^{p(x)}$ The norm in the space $\tilde{L}^{p(x)}$ is given by (1.30) while in $\mathcal{L}L^{p(x)} = \hat{L}^{p(x)}$ by (1.5).

1.7. One norm more in $L^{p(x)}(\Omega)$.

We introduce now another norm inspired by a norm known for the Orlicz spaces [6] . We put

$$\|f\|_p^\star = \inf_{\lambda>0} F(\lambda) \ , \quad F(\lambda) = \lambda \ + \ \lambda \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx \tag{1.40}$$

assuming that $|E_\infty| = 0$ for simplicity. The axioms for norms are easily verified.

Lemma 1.24. *The function* $c(p) = p(p-1)^{\frac{1}{p}-1}$, $1 \le p < \infty$, *is increasing for* $1 \le p \le 2$ *and decreasing for* $p \ge 2$ *and*

$$1 \le c(p) \le 2, \ 1 \le p \le \infty; \quad c(1) = 1, \ c(2) = 2, \ c(\infty) = 1.$$

Proof is direct.

Theorem 1.25. *Let* $|E_\infty| = 0$, $1 \le p_0 \le p(x) \le P < \infty$. *The norm (1.40) is equivalent to the norm* $\|f\|_p$ *:*

$$c_1\|f\|_p \le \|f\|_p^\star \le c_2\|f\|_p \tag{1.41}$$

where $c_1 = \min\{c(p_0), c(P)\} \ge 1$, *and* $c_2 = \max\{c(p_0), c(P)\}$,*if* $P \le 2$ *or* $p_0 \ge 2$, *and* $c_2 = 2$, *if* $p_0 \le 2 \le P$.

Proof. Evidently, $F(\|f\|_p) = 2\|f\|_p$ by (1.7). Hence $\|f\|_p^\star \le 2\|f\|_p$.

To prove the right-hand side inequality in (1.41) we note that

$$\|f\|_p^\star = \inf_{\lambda>0} F(\lambda\|f\|_p) \tag{1.42}$$

The evident inequality $F(\lambda\|f\|_p) \ge (\lambda + \lambda^{1-t})\|f\|_p$ is valid, where $t = p_0$ if $0 < \lambda \le 1$ and $t = P$ if $\lambda \ge 1$. So, from (1.42) we arrive at the left-hand side inequality in (1.41) with

$$c_1 = \min\left\{\min_{0<\lambda\le1}(\lambda + \lambda^{1-p_0}), \min_{\lambda\ge1}(\lambda + \lambda^{1-P}), \right\}.$$

An easy calculation and Lemma 1.24 give the value of this constant as indicated in the theorem. Similarly the right-hand side inequality in (1.41) is obtained if we take use of the inequality $F(\lambda\|f\|_p) \le (\lambda + \lambda^{1-\bar{t}})\|f\|_p$ where $\bar{t} = P$, if $\lambda \le 1$ and $\bar{t} = p_0$ if $\lambda \ge 1$. □

Theorem 1.26. *Let* $1 \le p(x) \le P < \infty$, $|E_\infty| = 0$ *and* $|\{x : p(x) > 1\}| > 0$. *Infinum in (1.40) is reached at* $\lambda = \lambda_0 = \|af\|_p$, *with* $a(x) = [p(x) - 1]^{\frac{1}{p(x)}}$, *so that*

$$\|f\|_p^\star = \|af\|_p \left\{ 1 \ + \ \int_\Omega \left| \frac{f(x)}{\|af\|_p} \right|^{p(x)} dx \right\}. \tag{1.43}$$

Proof. We have $\frac{dF(\lambda)}{d\lambda} = 1 \ - \ \int_\Omega [p(x) - 1] \left| \frac{f(x)}{\lambda} \right|^{p(x)} dx$, the differentiation under the integral sign being easily verified since $P < \infty$. The equation $dF(\lambda)/d\lambda = 0$ gives $\lambda = \|af\|_p$. □

Remark 1.27. *The equivalence of norms presented in Theorem 1.6 follows also from (1.43).*

1.8. Minkowski inequality.

Theorem 1.28 ([7]). *Let* $1 \le p(x) \le \infty$, $P < \infty$ *and* $p_0^1 > 1$. *Then*

$$\left\| \int_\Omega f(\cdot, y) dy \right\|_p^{**} \le \int_\Omega \|f(\cdot, y)\|_p^{**} dy \quad . \tag{1.44}$$

Corollary. *Let* $1 \le p(x) \le \infty$, $P < \infty$ *and* $p_0^1 > 1$. *Then*

$$\left\| \int_\Omega f(\cdot, y) dy \right\|_p^{*} \le c_1 \int_\Omega \|f(\cdot, y)\|_p^{*} dy \ , \tag{1.45}$$

$$\left\| \int_\Omega f(\cdot, y) dy \right\|_p \le c_2 \int_\Omega \|f(\cdot, y)\|_p dy \tag{1.46}$$

where $c_1 = 1$ *if* $|E_1| = 0$ *and* $c_1 = 2^{-1+Q/q_0^1}$ *otherwise, while* $c_2 = kc_1$ *if* $|E_\infty| = |E_1| = 0$ *and* $c_2 = 3kc_1$ *otherwise;* $k = \frac{1}{p_0} + \frac{1}{q_0}$.

Proof. The inequality (1.45) with $c_1 = 2^{-1+Q/q_0^1}$ follows from (1.44) in view of (1.33). Similarly , (1.46) follows from (1.44) by (1.34) and (1.33). To show that $c_1 = 0$ in (1.45) in case $|E_1| = 0$, we note that

$$\left\| \int_\Omega f(\cdot, y) dy \right\|_p^{*} \le \sup_{\rho_q(\varphi) \le 1} \int_\Omega \|\varphi\|_q \|f(\cdot, y)\|_p^{*} dy$$

and it remains to note that the conditions $\rho_q(\varphi) \le 1$ and $\|\varphi\|_q \le 1$ are equivalent in case $|E_1| = 0$ in view of (1.11)-(1.12). □

Remark 1.29. *Fubini theorem in the form*

$$\| \, \|f(x,y)\|_{p(y)}\|_{p(x)} = \| \, \|f(x,y)\|_{p(x)}\|_{p(y)},$$

valid for $p(x) = const$, is not in general true for a variable exponent, even in the form of the inequality

$$\| \, \|f(x,y)\|_{p(y)}\|_{p(x)} \le c\| \, \|f(x,y)\|_{p(x)}\|_{p(y)} \ . \tag{1.47}$$

Really, taking $n = 1$ for simplicity and $p(x) = p_1$ for $-1 < x < 0$, $p(x) = p_2$ for $0 < x < 1$, we can easily see that (1.47) would yield the equivalence of mixed norms in the spaces $L^{p_1}(L^{p_2}([0,1]))$, $L^{p_2}(L^{p_1}([0,1]))$, which is not true, as is known [1] , [2] .

II. Estimates of norms of power functions of distance, truncated to a ball of a radius $r > 0$.

2.1. Preliminaries.

In further applications, for example, to potential type operators in $L^{p(x)}(\Omega)$ we shall need an information about behaviour of norms $\|f_\beta\|$ where

$$f_\beta(x) = f_\beta(x_0, r; x) = \chi_r(x - x_0)|x - x_0|^{\beta(x_0)} \ , \ x_0 \in R^n,$$

where $\chi_r(x)$ is a characteristic function either of interior or exterior of the ball of the radius $r > 0$. To obtain these estimates, some minimal requirement of smoothness of $p(x)$ arises - we take it in the class w-Lip (R^n), see its definition below. There arises also a notion of norming value r_0 of radius for which $\|f_\beta\| = 1$. In case of $\beta = 0$, r_0 gives the ball of the volume 1 , while in the general case r_0 is as a root of some special equation.

Everywhere below in Section 2 it is assumed that

$$1 \le p_0 \le p(x) \le P < \infty \ , \ |E_\infty| = 0. \tag{2.1}$$

Definition 2.1. *A function $f(x)$ is said to belong to w-Lip(R^n), if $f(x) \in C(R^n)$ and $|f(x+h) - f(x)| \leq \frac{A}{\log \frac{1}{|h|}}$, $x \in R^n, h \in R^n, |h| < 1$ where $A > 0$ (the weak Lipschitz condition).*

Definition 2.2. *A function $f(x)$ is said to have the w-Lip - behaviour at infinity, if it is continuous at the infinite point and $|f(x) - f(\infty)| \log(e+|x|) \leq A_\infty < \infty$, where $A_\infty > 0$ does not depend on x .*

Definition 2.3. *A function $f(x)$ is said to belong to w-Lip $(\dot{R}^n)$, if $f(x) \in w - Lip(R^n) \bigcap C(\dot{R}^n)$ and has the $w - Lip$- behaviour at infinity .*

We first study the case of the interior of the ball : $\chi_r = \chi_{B(x_0,r)}(x)$, where $\chi_{B(x_0,r)}(x)$ is the characteristic function of the ball $B(x_0, r)$ centered at x_0 . . For brevity we denote

$$\lambda_\beta = \lambda_\beta(x_0, r) = \|f_\beta\|_p = \| \mid x - x_0 \mid^{\beta(x_0)} \chi_{B(x_0,r)}(x)\|_p.$$

According to (1.7) we have

$$\int_{|y| \leq r} \left(\frac{|y|^{\beta(x_0)}}{\lambda_\beta} \right)^{p(x_0+y)} dy = 1 \tag{2.2}$$

2.2. On convergence of integrals of power functions with a variable exponent .

Let

$$J = \int_{|x-x_0|<r} \frac{dx}{|x - x_0|^{a(x)}}. \tag{2.3}$$

We assume that $\sup_{|x-x_0|\leq 1} a(x) < +\infty$, $\inf_{|x-x_0|\geq 1} a(x) > -\infty$ (the latter in the case $r > 1$ only).

Lemma 2.6. *Let $a(x) \leq b(x)$, $|x - x_0| \leq \epsilon$, where $b(x) \in C\left(B(x_0, r)\right)$ and $b(x_0) < n$. Then the integral (2.3) converges.*

Proof is direct.

Corollary. *Let $x_0 \in \Omega \subset R^n$ and $|\Omega| < \infty$. Then $|x - x_0|^{-a(x)} \in L^{p(x)}(\Omega)$, $0 \leq p(x) < \infty$, if*

$$1) \quad \sup_{|x-x_0|\leq 1} a(x)p(x) < +\infty \ , \quad \inf_{|x-x_0|\geq 1} a(x)p(x) > -\infty$$

2) $a(x) \leq \frac{b(x)}{p(x)}$, $|x - x_0| < \epsilon$ *for some* $\epsilon > 0$, *where* $b(x) \in C\left(B(x_0, \epsilon)\right)$ *and* $b(x_0) < n$.

Lemma 2.7. *Let*

$$a(x) - a(x_0) \geq \frac{A}{\log|x - x_0|^{-1}} \ , \ A > 0 \ , \tag{2.4}$$

in some neighbourhood of the point x_0. Then the condition $a(x_0) < n$ is necessary for convergence of the integral (2.3).

Proof. The condition (2.4) is equivalent to the inequality $|x-x_0|^{a(x_0)-a(x)} \geq e^A > 0$, and then the necessity of the condition $a(x_0) < n$ becomes obvious .

2.3. The norming value of radius.

The function $\lambda_\beta(x_0, r)$ increases with respect to r . We are interested in separating the cases $\lambda_\beta(x_0, r) < 1$ and $\lambda_\beta(x_0, r) > 1$.

Definition 2.8. *The value $r = r_0 = r_0(x_0)$ of the radius is the norming one for the function $f_\beta(x_0, r; x)$ in the space $L^{p(x)}(R^n)$, if $\lambda_\beta(x_0, r_0) = 1$.*

It may happen that $r_0 = \infty$ or does not exist . Evidently,

$$\lambda_\beta(x_0, r) \leq 1 \Longleftrightarrow 0 < r \leq r_0, \tag{2.5}$$

$$\lambda_\beta(x_0, r) \geq 1 \Longleftrightarrow r \geq r_0, \tag{2.6}$$

We introduce the function

$$F(x_0) = \int_{R^n} |x|^{\beta(x_0)p(x+x_0)} dx, \quad x_0 \in R^n, \tag{2.7}$$

Lemma 2.9. *Let w-Lip $(\dot{R}^n)$ and $|\beta(x_0)| < \infty$. Then $F(x_0) < \infty$, if and only if*

$$n + \beta(x_0)p(x_0) > 0 \ , \quad n + \beta(x_0)p(\infty) < 0. \tag{2.8}$$

Proof. We have $F(x_0) = \int_{|x|<1} |x|^{\beta(x_0)p(x+x_0)} dx + \int_{|x|>1} |x|^{\beta(x_0)p(x+x_0)} dx = G(x_0) + H(x_0)$, so that the convergence of $f(x_0)$ is equivalent to the simultaneous convergence of $G(x_0)$ and $H(x_0)$. Obviously, $G(x_0) = \int_{|x|>1} |x|^{\beta(x_0)p(x_0)} g(x) dx$ and $H(x_0) = \int_{|x|>1} |x|^{\beta(x_0)p(\infty)} h(x) dx$, where

$$g(x) = |x|^{\beta(x_0)[p(x+x_0)-p(x_0)]} \ , \tag{2.9}$$

$$h(x) = |x|^{\beta(x_0)[p(x+x_0)-p(\infty)]} \ .$$

Since $|x| \leq 1$ in the first integral, in accordance with Definition 2.1 we have

$$e^{-A|\beta(x_0)|} \leq g(x) \leq e^{A|\beta(x_0)|}. \tag{2.10}$$

Similarly, $e^{-A_\infty|\beta(x_0)|} \leq h(x) \leq e^{A_\infty|\beta(x_0)|}$ in accordance with Definition 2.2. Therefore, convergence of the integral $G(x_0)$ is equivalent to the first of the conditions (2.8), while that of $H(x_0)$ is equivalent to the second one. □

We shall also need a more strong requirement for the first condition in (2.8) to be uniform in x_0 :

$$d \ := \inf_{x\in R^n}\left[1+\frac{\beta(x)p(x)}{n}\right] > 0. \qquad (2.11)$$

Remark 2.10. *The condition (2.11) arises only in case of $\beta < 0$. In the case when $\beta(x) \geq 0$ a.e. in R^n we make the convention that $d = 1$.*

Lemma 2.11. *The norming value r_0 of radius is the root of the equation*

$$\int_{|x|<r_0} |x|^{\beta(x_0)p(x+x_0)}dx = 1. \qquad (2.12)$$

It does exist if $p(x) \in$ w-Lip $(R^n), n+\beta(x_0)p(x_0) > 0$ and $F(x_0) > 1$.

Proof. By (2.2) , $\lambda_\beta = 1$ is just the same as (2.12). Since the left-hand side in (2.12) increases with respect to r_0, the equation does have a unique root, if only the integral converges. This is the case by Lemma 2.7. □

Lemma 2.12. *Let $p(x) \in$ w-Lip (R^n) . If $r_0 \leq 1$, then*

$$e^{-A|\beta(x_0)|} \leq \int_{|x|<r_0} |x|^{\beta(x_0)p(x_0)}dx \leq e^{A|\beta(x_0)|}. \qquad (2.13)$$

Proof. From (2.12) we have $\int_{|x|<r_0} |x|^{\beta(x_0)p(x_0)}g(x)dx = 1$ where $g(x)$ is the function from (2.9). Applying (2.10), we obtain (2.13). □

The next subsections 2.4 and 2.5 are devoted to the proof of the uniform boundedness of r_0 as a function of x_0 : $0 < c \leq r_0(x_0) \leq C < \infty$ with c and C not depending on x_0 .

2.4. Lower bound for $r_0(x_0)$.

Besides (2.11) we shall also deal with the condition

$$d_1 = \inf_{x\in R^n}\left(1+\frac{\beta(x)P}{n}\right) > 0 \ . \qquad (2.14)$$

Similarly to Remark 2.10 we take $d_1 = 1$ if $\beta(x) \geq 0$ a.e.

Lemma 2.13. *Let $d > 0$ and $p(x) \in$ w-Lip (R^n). Then*

$$r_0 \geq e^{-\frac{A}{p_0}\max\{\frac{1-d}{d},1\}}\left[\min\left(1,\frac{d}{|B_n|}\right)\right]^{\frac{1}{nd}}, \qquad (2.15)$$

where A is the constant from the Definition 2.1 of the w-Lip - condition for the function $p(x)$. If $d_1 \geq 0$, then

$$r_0 \geq \left[\min\left(1, \frac{d}{|B_n|}\right)\right]^{\frac{1}{nd_1}}, \tag{2.16}$$

without the assumption $p(x) \in$ w-Lip (R^n) .

Proof. We suppose that $r_0 = r_0(x_0) \leq 1$ (otherwise the estimate (2.16) is already valid). From (2.12) we have

$$1 \leq \int_{|x|<r_0} |x|^{\beta(x_0)q} dx \tag{2.17}$$

where $q = p_0$, if $\beta(x_0) \geq 0$ and $q = P$ if $\beta(x_0) < 0$. Denoting $1 + \frac{\beta q}{n} = t(\geq d_1)$, we obtain $r_0 \geq \left(\frac{t}{|B_n|}\right)^{\frac{1}{nt}}$. Hence (2.16) follows, since

$$\inf_{d_1 \leq t < \infty} \left(\frac{t}{|B_n|}\right)^{\frac{1}{t}} = \min\left(1, \frac{d_1}{|B_n|}\right)^{\frac{1}{d_1}} . \tag{2.18}$$

Let $d > 0$. Taking $r_0 \leq 1$ again , from the left-hand side inequality in (2.13) we derive the estimate $r_0^{n+\beta(x_0)p(x_0)} \geq [n + \beta(x_0)p(x_0)]\,|S_{n-1}|^{-1}e^{-A|\beta(x_0)|}$. Hence

$$r_0 \geq [k(t)]^{\frac{1}{n}} \tag{2.19}$$

where $t = 1 + \frac{\beta(x_0)p(x_0)}{n} \geq d > 0$ and $k(t) = \left(\frac{t}{|B_n|}\right)^{\frac{1}{t}} e^{-\frac{nA}{p(x_0)}\frac{|t-1|}{t}}$. Taking (2.18) and the equality

$$\sup_{t>d} \frac{|t-1|}{t} = \max\left(1, \frac{1-d}{d}\right) \tag{2.20}$$

into account, we see that (2.15) follows from (2.19). □

2.5 Upper bound for $r_0(x_0)$.

Similarly to case of the lower bound, we give the upper bound without assumption on w-Lip - behaviour of the function $p(x)$ in case $d_1 > 0$.

Lemma 2.14. *Let $d_1 > 0$. Then*

$$r_0(x_0) \leq e^{\frac{1+d_1|B_n|^{-1}}{ned_1}} \tag{2.21}$$

Proof. Assuming that $r_0 \geq 1$, from (2.12) we obtain $1 \geq \int_{1<|x|<r_0} |x|^{\beta(x_0)p(x+x_0)}dx \geq \int_{1<|x|<r_0} |x|^{\beta(x_0)q}dx$, where q is the same as in (2.17). Hence

$$r_0(x_0) \leq \left(1 + \frac{t}{|B_n|}\right)^{\frac{1}{nt}} \tag{2.22}$$

with $t = 1 + \frac{\beta(x_0)q}{n} \geq d_1$. Since $(1+at)^{\frac{1}{t}} \leq e^{\frac{1+ab}{be}}$, $t \geq b > 0$, $a > 0$, we see that (2.21) follows from (2.22).

Lemma 2.15. *Let $d > 0$ and let $p(x) \in$ w-Lip (R^n). Then*

$$r_0(x_0) \leq R_0 \ := e^{\frac{1}{e|B_n|} + \frac{A}{p_0}\max(1, \frac{1-d}{d})} . \tag{2.23}$$

Proof. In accordance with Lemma 2.11 we shall show that there exist a number $R_0 > 0$ not depending on x_0 such that

$$G(R_0, x_0) \ := \int_{|x|<R_0} |x|^{\beta(x_0)p(x+x_0)}dx \geq 1 \tag{2.24}$$

for all $x_0 \in R^n$. We have $G(R_0, x_0) = \int_{|x|<R_0} |x|^{\beta(x_0)}g(x)dx$ where $g(x)$ is defined in (2.9). Hence, by (2.13)

$$G(R_0, x_0) \ \geq \ \frac{|B_n|e^{-A|\beta(x_0)|}}{1 + \frac{\beta(x_0)p(x_0)}{n}} R_0^{n+\beta(x_0)p(x_0)}.$$

Therefore, to have (2.24) , it is sufficient to choose R_0 in such a way that $R_0^{n+\beta(x_0)p(x_0)} \geq \frac{1}{|B_n|}e^{A|\beta(x_0)|}\left(1 + \frac{\beta(x_0)p(x_0)}{n}\right)$ or $R_0 \geq \left[\left(\frac{t}{|B_n|}\right)^{\frac{1}{t}} e^{\frac{A|t-1|}{t}\frac{n}{p(x_0)}}\right]^{\frac{1}{n}}$, $t = 1 + \frac{\beta(x_0)p(x_0)}{n}$. By (2.20) we may take $R_0 \ \geq \ e^{\frac{A}{p_0}\max(1, \frac{1-d}{d})}\left(\frac{t}{|B_n|}\right)^{\frac{1}{nt}}$. Since $\max_{t>0}(at)^{1/t} = e^{a/e}$, this gives (2.23). □

2.6 Estimates of the norm $\|f_\beta\|_p$ as $r \to 0$.

Lemma 2.16. *The norm λ_β is estimated by*

$$[J_\beta(r)]^{\frac{1}{p(x_0,r)}} \leq \lambda_\beta \leq [J_\beta(r)]^{\frac{1}{P(x_0,r)}}, \quad 0 \leq r \leq r_0, \tag{2.25}$$

where

$$J_\beta(r) = J_\beta(x_0, r) = \int_{|x|<r} |x|^{\beta(x_0)p(x+x_0)}dx. \tag{2.26}$$

Proof. From (2.2) and (2.5) we have

$$\frac{1}{\lambda^{p(x_0,r)}}\int_{|x|<r}|x|^{\beta(x_0)p(x+x_0)}dx \le 1 \le \frac{1}{\lambda^{P(x_0,r)}}\int_{|x|<r}|x|^{\beta(x_0)p(x+x_0)}dx$$

which coincides with (2.25).

Lemma 2.17. *The integral $J_\beta(r)$ satisfies the estimates*

$$k(r)\left(\frac{r}{r_0}\right)^{n+\beta q_1} \le J_\beta(r) \le k(r)\left(\frac{r}{r_0}\right)^{n+\beta q_2}, \quad 0<r\le r_0, \tag{2.27}$$

where $q_1 = P(x_0,r)$ if $\beta(x_0)\ge 0$ and $q_1 = p(x_0,r)$, if $\beta(x_0)\le 0$, while $q_2 = p(x_0,r)$ if $\beta(x_0)\ge 0$ and $q_2 = P(x_0,r)$, if $\beta(x_0)\le 0$ and

$$k(r) = \int_{|x|<r_0}|x|^{\beta(x_0)p(\frac{r}{r_0}x+x_0)}dx\ . \tag{2.28}$$

Proof. The change of variables $x = \frac{r}{r_0}y$ yields $J_\beta(r) = \left(\frac{r}{r_0}\right)^n\int_{|y|<r_0}\left(\frac{r}{r_0}|y|\right)^{\beta(x_0)p(\frac{r}{r_0}y+x_0)}dy$. Hence (2.27) is easily derived. □

We shall see below that $0<c_1\le k(r)=k(x_0,r)\le c_2<\infty$ uniformly in x_0 under the appropriate assumptions on $p(x)$ and $\beta(x)$.

Lemma 2.18 . (Estimates for a fixed point x_0). *Let $p(x)\in w$-Lip (R^n) and $n+\beta(x_0)p(x_0)>0$. Then*

$$0<k_-\le k(r)\le k_+<\infty \tag{2.29}$$

where

$$k_- = A_n^-(x_0)\min\left(1, r_0^{n+\beta(x_0)p(x_0)}\right),\ \ k_+ = A_n^+(x_0)\max\left(r_0^{n+\beta(x_0)q}, r_0^{n+\beta(x_0)p(x_0)}\right),$$

with $q=P$ if $\beta(x_0)\ge 0$ and $q=p_0$ if $\beta(x_0)<0$ and

$$A_n^{\pm}(x_0) = \frac{e^{\pm A|\beta(x_0)|}|S_{n-1}|}{n+\beta(x_0)p(x_0)}. \tag{2.30}$$

Proof. Let $0<r_0\le 1$ first . We have

$$k(r) = \int_{|x|<r_0}|x|^{\beta(x_0)p(x_0)}h_r(x)dx \tag{2.31}$$

where $h_r(x) = |x|^{\beta(x_0)[p(\frac{r}{r_0}x+x_0)-p(x_0)]}$. Since $p(x) \in$ w-Lip (R^n) and $\left|\frac{r}{r_0}y\right| \leq r \leq r_0 \leq 1$, we have

$$|\log h_r(x)| \leq A|\beta(x_0)| \left|\frac{\log|x|}{\log\frac{r_0}{r|x|}}\right| = A|\beta(x_0)| \frac{\log\frac{1}{|x|}}{\log\frac{1}{|x|} + \log\frac{r_0}{r}} \leq A|\beta(x_0)|.$$

Then

$$e^{-A|\beta(x_0)|} \leq h_r(x) \leq e^{A|\beta(x_0)|} , \tag{2.32}$$

so that the estimates (2.29) follow from (2.31)

Let now $r_0 > 1$. We have

$$k(r) = \int_{|x|<1} |x|^{\beta(x_0)p(x_0)} h_r(x)dx + \int_{1\leq|x|\leq r_0} |x|^{p(\frac{r}{r_0}x+x_0)}dx . \tag{2.33}$$

In the first term the function $h_r(x)$ is estimated in the same way as in the previous case. So, (2.33) implies

$$k(r) \leq A_n^+(x_0) + \frac{|S_{n-1}|}{n+q\beta(x_0)} \left[r_0^{n+q\beta(x_0)} - 1\right] \leq A_n^+(x_0) r_0^{n+q\beta(x_0)} \tag{2.34}$$

since $n + q\beta(x_0) \geq n + \beta(x_0)p(x_0)$.

Similarly to (2.34) the lower bound

$$k(r) \geq \int_{|x|<1} |x|^{\beta(x_0)p(x_0)} h_r(x)dx \geq A_n^-(x_0),$$

for $0 < r \leq r_0,\ \ R_0 \geq 1$, can be obtained. Gathering the estimates we arrive at (2.29)

Lemma 2.19. (The uniform estimate). *Let* $p(x) \in$ *w-Lip* (R^n), $\gamma := \sup_{x\in R^n} \beta(x) < +\infty$ *and let* $d > 0$. *Then* $0 < c_- \leq k(r) \leq c_+ < \infty$, $0 < r \leq r_0$, *with* c_+ *and* c_- *not depending on* r *and* x_0; *in the case* $r_0 \leq 1$ *one may take* $c_\pm = e^{\pm 2AB}$ *where* $B = \sup|\beta(x)| \leq \max\left(\gamma, \frac{n(1-d)}{d}\right)$.

Proof. Let $r_0 \leq 1$. From (2.31)-(2.32) we obtain

$$e^{-A|\beta(x_0)|} \int_{|x|<r_0} |x|^{\beta(x_0)p(x_0)}dx \leq k(r) \leq e^{A|\beta(x_0)|} \int_{|x|<r_0} |x|^{\beta(x_0)p(x_0)}dx$$

Hence by Lemma 2.12 $e^{-2A|\beta(x_0)|} \leq k(r) \leq e^{2A|\beta(x_0)|}$ which proves the lemma for $r_0 \leq 1$. If $r_0 > 1$, the uniform estimates follow from (2.82) in view of the uniform estimates (2.15) and (2.23) for $r_0(x)$. □

Lemma 2.20. *Let $p(x) \in$ w-Lip (R^n) and $d > 0$. There exists a constant $c > 0$, not depending on r and x_0 such that for all $0 < r \leq r_0$ we have*

$$|P(x_0, r) - p(x_0)| \log \frac{r_0}{r} \leq c \ , \ |p(x_0, r) - p(x_0)| \log \frac{r_0}{r} \leq c, \tag{2.35}$$

Proof. Let $r \leq 1$. Since $p(x) \in$ w-lip (R^n), we have

$$\sup_{|x-x_0|\leq r} \left(| \ p(x) - p(x_0) \ | \log \frac{1}{| \ x - x_0 \ |} \right) \leq A, \quad 0 < r \leq 1 \ .$$

Then, moreover, $\left|\sup_{|x-x_0|\leq r}[p(x) - p(x_0)]\right| \log \frac{1}{r} \leq A$. The latter is nothing else but $[P(x_0, r) - p(x_0)] \log \frac{1}{r} \leq A$. Hence the first of the inequalities (2.35) follows with $c = A + (P - p_0) \sup_{x_0} |\log r_0|$. If $r \geq 1$, the inequalities (2.33) are trivial . It remains to note that $\sup_{x_0} |\log r_0| < \infty$ by Lemmas 2.13 and 2.14.

To obtain the second of the inequalities in (2.35), we remark that $|\inf f(x)| \leq \sup |f(x)|$ and then from what was above it follows that $| \inf_{|x-x_0|\leq r}[p(x) - p(x_0)] \log \frac{1}{r} \leq A$. Then the same arguments give the required estimate. □

Theorem 2.21. *Let $p(x) \in$ w-Lip (R^n), $d > 0$ and $\sup_x |\beta(x)| < \infty$. Then*

$$c_1 \left(\frac{r}{r_0}\right)^{\frac{n}{p(x_0)} + \beta(x_0)} \leq \lambda_\beta(x_0, r) \leq c_2 \left(\frac{r}{r_0}\right)^{\frac{n}{p(x_0)} + \beta(x_0)}, \ 0 < r \leq r_0. \tag{2.36}$$

Proof. From (2.25), (2.27) and (2.29) we conclude that

$$c_1 \left(\frac{r}{r_0}\right)^{\frac{n}{p(x_0,r)} + \beta(x_0)} \leq \lambda_\beta \leq c_2 \left(\frac{r}{r_0}\right)^{\frac{n}{P(x_0,r)} + \beta(x_0)}, \tag{2.37}$$

in case $\beta(x_0) \leq 0$ and

$$c_1 \left(\frac{r}{r_0}\right)^{\frac{n}{p(x_0,r)} + \beta(x_0) \frac{P(x_0,r)}{p(x_0,r)}} \leq \lambda_\beta \leq c_2 \left(\frac{r}{r_0}\right)^{\frac{n}{P(x_0,r)} + \beta(x_0) \frac{p(x_0,r)}{P(x_0,r)}}, \tag{2.38}$$

in case $\beta(x_0) \geq 0$, c_1 and c_2 not depending on r and x_0. Hence we derive (2.36) by means of Lemma 2.20. □

Lemma 2.22. *Under the assumptions of Lemma 2.19 , $\lambda_\beta(x_0, r) \leq c < \infty$ for $0 < r \leq R$, with c not depending on r and x_0 .*

Proof. Because of (2.35) we take $r \geq r_0$. Since $\lambda_\beta \geq 1$ in this case , from (2.2) and (2.11) we obtain $1 \leq \frac{1}{\lambda_\beta^{p_0}} \int_{|x|<r} |x|^{\beta(x_0)p(x+x_0)} dx =$ $\frac{1}{\lambda_\beta^{p_0}} \left[1 + , \int_{r_0 \leq |x| < 1} |x|^{\beta(x_0)p(x+x_0)} dx + \int_{1 \leq |x| < r} |x|^{\beta(x_0)p(x+x_0)} dx \right]$ where the splitting of the integral over the layer $r_0 \leq |x| \leq r$ to two integrals should be omitted in case of $r_0 \geq 1$. Since $\sup_x \beta(x) < +\infty$, $r_0(x_0)$ is separated from zero, so that the expression in the brackets is easily estimated for $0 \leq r \leq R$ by a constant owing to the w-Lip - condition. □

Corollary. *Under the assumptions of Theorem 2.21*

$$c_3 \ r^{\frac{n}{p(x_0)} + \beta(x_0)} \leq \lambda_\beta(x_0, r) \leq c_4 \ r^{\frac{n}{p(x_0)} + \beta(x_0)}, \ 0 < r \leq R,$$

with c_3 *and* c_4 *depending* R *only.*

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