Weighted Sobolev theorem
in Lebesgue spaces with variable exponent: corrigendum

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Abstract

We fill in a gap discovered in the proof of Theorem A, on weighted Sobolev type boundedness for potential operators in variable exponent Lebesgue spaces, in the paper of the authors "Weighted Sobolev theorem in Lebesgue spaces with variable exponent", J. Math. Anal. and Applic., 2007, vol. 335, No 1, 560-583. The proof remains the same in the case where the Matuszewska-Orlich indices $m(w)$ and $M(w)$ of the weight w are both positive or negative, but in the case where they have different signs, the proof needs some additional arguments and requires a slightly different formulation of the result.

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1 Introduction

Let Ω be an open set in $\mathbb{R}^n$, $p : \Omega \to [1, \infty)$ a measurable function on Ω with $\inf\limits_{x\in\Omega} p(x) > 1$, $\sup\limits_{x\in\Omega} p(x) < \infty$ and

$$L^{p(\cdot)}(\Omega, \rho) = \left\{ f : \rho(x)|f(x)|^{p(x)} \in L^1(\Omega) \right\},$$

where $\rho(x) = w(|x - x_0|)$ with $x_0 \in \overline{\Omega}$. We assume that w is in the generalized Bary-Stechkin-type class, Definition 2.2 in [1]. We refer also to Definition 2.3 there on Matuszewska-Orlich indices $m(w)$ and $M(w)$ of the weight. Recall that

$$-\infty < m(w) \leq M(w) < \infty$$

for weights in such class. In [1], within the frameworks of the spaces $L^{p(\cdot)}(\Omega, \rho)$, we studied the potential type operator

$$I^{\alpha(\cdot)} f(x) = \int\limits_{\Omega} \frac{f(y)\, dy}{|x - y|^{n-\alpha(x)}}, \qquad x \in \Omega. \tag{1.1}$$

of variable order $\alpha(x)$, where $\inf\limits_{x\in\Omega} \alpha(x) > 0$. The paper [1] contains the following theorem.

Theorem A. *Let $p(x)$ and $\alpha(x)$ satisfy local log-condition in Ω and the condition $\sup\limits_{x\in\Omega} \alpha(x)p(x) < n$. If the indices $m(w)$ and $M(w)$ of the weight satisfy the condition*

$$\alpha(x_0)p(x_0) - n < m(w) \le M(w) < n[p(x_0) - 1]. \tag{1.2}$$

Then

$$\left\|I^{\alpha(\cdot)}f\right\|_{L^{q(\cdot)}\left(\Omega, w^{\frac{q}{p}}(|x-x_0|)\right)} \le C\,\|f\|_{L^{p(\cdot)}(\Omega, w(|x-x_0|))}\,. \tag{1.3}$$

However, the proof of this theorem given in [1] contains a gap. We correct the proof. This correction led to a certain modification of the statement. Namely, the statement of Theorem A and its proof remain without changes when the indices are both positive or negative:

$$\alpha(x_0)p(x_0) - n < m(w) \le M(w) < 0 \quad \text{or} \quad 0 < m(w) \le M(w) < n[p(x_0) - 1], \tag{1.4}$$

while in the case of different signs:

$$\alpha(x_0)p(x_0) - n < m(w) \le 0 \le M(w) < n[p(x_0) - 1], \tag{1.5}$$

the correction of the proof led to some modification of the weight on the left-hand side of inequality (1.3). The corrected version of Theorem A runs as follows.

Theorem A$_{corr}$. *Let $p(x)$ and $\alpha(x)$ satisfy local log-condition in Ω and the condition $\sup_{x\in\Omega} \alpha(x)p(x) < n$. If the indices $m(w)$ and $M(w)$ of the weight satisfy the condition* (1.4)*, then inequality* (1.3) *is valid. If* (1.5) *holds, then*

$$\left\|I^{\alpha(\cdot)}f\right\|_{L^{q(\cdot)}\left(\Omega, \varphi(|x-x_0|)w^{\frac{q}{p}}(|x-x_0|)\right)} \le C\,\|f\|_{L^{p(\cdot)}(\Omega, w(|x-x_0|))}\,, \tag{1.6}$$

where $\varphi(t)$ is any bounded weight function such that $\int\limits_0^\ell \frac{\varphi(t)}{t}\,dt < \infty,\ \ell = diam\,\Omega$.

2 Proof of Theorem A$_{corr}$

2.1 The case (1.4); the proof contained in [1]

We start with the part which does not need changes, to underline some points. As in [1], we take $x_0 = 0$. First we note that estimate (5.8) in [1] may be rewritten in the form

$$|\mathbb{B}_r(x)| \le C\, r^{-\frac{n}{q(x)}}[w(r+x)]^{-\frac{1}{p(x)}}, \tag{2.7}$$

where we replaced $r_x = \max\{r, |x|\}$ by $r + |x|$, which is possible when w has finite indices $m(w)$ and $M(w)$, since $r_x \le r + |x| \le 2r_x$. Therefore, inequality (5.9) in [1] holds in the form

$$I^{\alpha(\cdot)}f(x) \le C\left[r^{\alpha(x)}\mathcal{M}f(x) + [w(r+|x|)]^{-\frac{1}{p(x)}}\ r^{-\frac{n}{q(x)}}\right] \tag{2.8}$$

in all the cases, that is, independently of the signs of the indices $m(w)$ and $M(w)$. By the almost monotonicity properties of $w(r)$, from (2.8) it follows that

$$I^{\alpha(\cdot)}f(x) \leq C\left[r^{\alpha(x)}\mathcal{M}f(x) + [w(|x|)]^{-\frac{1}{p(x)}}\, r^{-\frac{n}{q(x)}}\right] \quad \text{when} \quad m(w) > 0, \tag{2.9}$$

because functions w with positive index $m(w)$ are almost increasing, see for instance [1], Theorem 2.4. Then all the arguments remain the same as in [1] after formula (5.9) there. This covers the case $0 < m(w) \leq M(w) < n[p(0) - 1]$. The case $\alpha(0)p(0) - n < m(w) \leq M(w) < 0$ is covered by the standard dual arguments, as on pages 575-576 of [1] in section 2^0 of the proof.

We recall that the minimizing value of r for the right-hand side of (2.9) is

$$r = r_0 := [w(|x|)]^{-\frac{1}{n}}\,[\mathcal{M}f(x)]^{-\frac{p(x)}{n}}. \tag{2.10}$$

Note that

$$r_0 \leq |x| \Longleftrightarrow \mathcal{M}f(x) \geq v(x), \tag{2.11}$$

$$r_0 \geq |x| \Longleftrightarrow \mathcal{M}f(x) \leq v(x), \tag{2.12}$$

where

$$v(x) = |x|^{-\frac{n}{p(x)}}[w(|x|)]^{-\frac{1}{p(x)}}$$

Note also that $w(|x|)[v(x)]^{p(x)} = \frac{1}{|x|^n}$, so that $v \notin L^{p(\cdot)}(\Omega, w)$. This means that the possibility (2.11) in fact cannot happen.

3 The case (1.4); the added proof

We transform the right-hand-side of (2.8) as follows

$$r^{-\frac{n}{q(x)}}[w(r + |x|)]^{-\frac{1}{p(x)}} = r^{-\frac{n}{q(x)}}[w(r + |x|)(r + |x|)^a]^{-\frac{1}{p(x)}}(r + |x|)^{\frac{a}{p(x)}},$$

where a is a number from formula (5.2) in [1], $a < n$. Since $w(t)t^a$ is almost increasing, we obtain

$$r^{-\frac{n}{q(x)}}[w(r + |x|)]^{-\frac{1}{p(x)}} \leq r^{-\frac{n}{q(x)}}[w(|x|)]^{-\frac{1}{p(x)}}\left(\frac{r + |x|}{|x|}\right)^{\frac{a}{p(x)}}.$$

Therefore, from (2.8) we obtain

$$I^{\alpha(\cdot)}f(x) \leq C\left(r^{\alpha(x)}\mathcal{M}f(x) + r^{-\frac{n}{q(x)}}[w(|x|)]^{-\frac{1}{p(x)}}\right) \quad \text{in the case where} \quad r \leq |x| \tag{3.13}$$

and

$$I^{\alpha(\cdot)}f(x) \leq C\left(r^{\alpha(x)}\mathcal{M}f(x) + r^{\frac{a}{p(x)}-\frac{n}{q(x)}}|x|^{-\frac{a}{p(x)}}[w(|x|)]^{-\frac{1}{p(x)}}\right) \quad \text{in the case where} \quad r \geq |x| \tag{3.14}$$

The minimizing value of $r = r_0$ for the right-hand side of(3.13) is given in (2.10). The minimizing value r_1 for (3.14) (obtained as the value of r for which both terms in (3.14) coincide), is

$$r_1 := |x|^{\frac{a}{a-n}}[w(|x|)]^{\frac{1}{a-n}} Mf(x)^{\frac{p(x)}{a-n}}. \tag{3.15}$$

Observe that

$$\frac{r_1}{|x|} = \left(\frac{r_0}{|x|}\right)^{\frac{n}{n-a}},$$

so that for r_1 we have exactly the same relations as in (2.11)-(2.12):

$$r_1 \le |x| \Longleftrightarrow \mathcal{M}f(x) \ge v(x), \tag{3.16}$$

$$r_1 \ge |x| \Longleftrightarrow \mathcal{M}f(x) \le v(x). \tag{3.17}$$

Therefore, from (3.13)-(3.14) we have

$$I^{\alpha(\cdot)}f(x) \le C r_0^{\alpha(x)} \mathcal{M}f(x) \quad \text{in the case where } \mathcal{M}f(x) \ge v(x) \tag{3.18}$$

and

$$I^{\alpha(\cdot)}f(x) \le C r_1^{\alpha(x)} \mathcal{M}f(x) \quad \text{in the case where } \mathcal{M}f(x) \le v(x). \tag{3.19}$$

Substituting the values of r_0 and r_1, we obtain

$$I^{\alpha(\cdot)}f(x) \le C[w(|x|)]^{-\frac{\alpha(x)}{n}} [Mf(x)]^{\frac{p(x)}{q(x)}} .$$

and

$$I^{\alpha(\cdot)}f(x) \le C|x|^{a-n}[w(x)]^{\frac{\alpha(x)}{a-n}} \mathcal{M}f(x)^{\frac{p_1(x)}{q(x)}}$$

respectively, where

$$p_1(x) = p(x)\left(1 - \frac{a\alpha(x)q(x)}{n(n-a)}\right) < p(x).$$

Consequently,

$$\int_\Omega [w(|x|)]^{\frac{q(x)}{p(x)}} \left|I^{\alpha(\cdot)}f(x)\right|^{q(x)} \, dx \le C \int_\Omega w(|x|)|\mathcal{M}f(x)|^{p(x)} \, dx.$$

in the first case, and

$$\int_\Omega [w(|x|)]^{\frac{q(x)}{p(x)}} \left|I^{\alpha(\cdot)}f(x)\right|^{q(x)} \, dx \le C \int_\Omega \frac{w^{\beta_1(x)}(|x|)}{|x|^{\beta_2(x)}}|\mathcal{M}f(x)|^{p_1(x)} \, dx, \tag{3.20}$$

in the second case, where

$$\beta_1(x) = q(x)\left[\frac{1}{p(x)} - \frac{\alpha(x)}{n-a}\right], \quad \beta_2(x) = \frac{a\alpha(x)q(x)}{n-a}.$$

There is nothing to do in the first case, so we have to work with inequality (3.20).

Let $p_2(x) = \frac{p(x)}{p_1(x)}$. Obviously, $\inf_{x\in\Omega} p_2(x) > 1$. Observe that with this notation we have

$$\beta_1(x) = \frac{1}{p_2(x)}, \quad \beta_2(x) = \frac{n}{p_2'(x)}.$$

An application of the weighted variable exponent Hölder inequality in (3.20) with the exponents $p_2(x)$ and $p_2'(x)$ is not helpful, if we wish to obtain the final inequality in form (1.3). Indeed, we have

$$\left\| \frac{[w(|x|)]^{\beta_1(x)-\frac{p_1(x)}{p(x)}}}{|x|^{\beta_2(x)}} \right\|_{L^{p_2'}} = \left\| \frac{1}{|x|^{\beta_2(x)}} \right\|_{L^{p_2'}} = \infty,$$

since $\beta_2(x)p_2'(x) \equiv n$. This explains the appearance of the additional factor φ in the weight in our proof. Instead of (3.20) we write

$$\int\limits_{\Omega} \varphi(|x|)[w(|x|)]^{\frac{q(x)}{p(x)}} \left|I^{\alpha(\cdot)}f(x)\right|^{q(x)} dx \leq C \int\limits_{\Omega} \frac{\varphi(|x|)[w(|x|)]^{\beta_1(x)}}{|x|^{\beta_2(x)}} |\mathcal{M}f(x)|^{p_1(x)} dx. \quad (3.21)$$

Then the Hölder inequality with the exponents $p_2(x)$ and $p_2'(x)$, the boundedness of the maximal operator in the space $L^{p(\cdot)}(\Omega, w)$ (see Theorem 2.9 in [1]), and the fact that $\beta_1(x) - \frac{p_1(x)}{p(x)} = 0$ provide inequality (1.6), if $\left\| \frac{\varphi(|x|)}{|x|^{\beta_2(x)}} \right\|_{L^{p_2'}} < \infty$. The latter is equivalent to $\int\limits_{\Omega} \frac{[\varphi(|x|)]^{p_2'(x)}}{|x|^n} dx < \infty$. Since $p_2'(x) > 1$ and φ is bounded, the condition $\int\limits_0^{\ell} \frac{\varphi(t)}{t}\, dt < \infty$ is sufficient for the latter.

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References

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