Proc. Razmadze Math. Institute
vol. 124, 2000, 73-106
MR 2002e:45007

N.K.Karapetiants and S.G.Samko

Singular integral equations on the real line with a fractional-linear Carleman shift

1 Introduction

Let $\tau(x) = \frac{\delta x+\beta}{x-\delta}$ be a fractional linear shift on R^1 satisfying the Carleman condition $\tau[\tau(x)] \equiv x$ and let

$$S\varphi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(t)dt}{t-x}.$$

We consider Fredholmness (=Noetherity) of singular integral operator with a fractional-linear shift:

$$K\varphi = a(x)\varphi(x) + b(x)\varphi[\tau(x)] + c(x)(S\varphi)(x) + d(x)(S\varphi)[\tau(x)], \quad x \in R^1 \qquad (A)$$

in the case of continuous coefficients. As is well known, the shift operator $Q\varphi = \varphi[\tau(x)]$ is not bounded in $L_p(R^1)$. Fredholm properties of the operator K are easily investigated in the special weighted space $L_p^{\frac{p}{2}-1}(R^1)$ with the weight function depending on p, see [5] or the book [6]. In that space the operators of the form (A) generate an algebra modulo compact operators.

However, it is of importance to know Fredholm properties of the operator (A) in the weighted space

$$L_p^{\gamma}(R^1) = \left\{\varphi : \int_{-\infty}^{\infty} |x-\delta|^{\gamma}|\varphi(x)|^p\, dx < \infty\right\},$$

when the weight "fixed" to the singular point $x = \delta$ of the shift $\tau(x)$ has the exponent $\gamma \in (-1, p-1)$ not depending on p. Such a necessity is caused not only by a natural desire to have an information about solvability of equations in the non-weighted case $\gamma = 0$, but also by the fact that in applications the special choice $\gamma = \frac{p}{2} - 1$ proves to be restrictive, see applications to potential operators with shifts in [16] and [6].

Such a modification of the setting of the problem, from $L_p^{\frac{p}{2}-1}(R^1)$ to $L_p^{\gamma}(R^1)$, seeming slight from the first point of view, radically changes the matter. In the space $L_p^{\frac{p}{2}-1}(R^1)$, the Fredholm theory of the equation (A) is of nature typical for singular integral equations with Carleman shift on a bounded curve. In the general case of the space $L_p^\gamma(R^1)$, the Fredholmness conditions prove to be more complicated, see for example Theorems 3.13 and 3.15, including an "additional" condition, see (3.81), (3.82) or (3.72). In this more general case the operator K reduces not to just a singular operator, as it happens in the case of the space $L_p^{\frac{p}{2}-1}(R^1)$, but to such an operator perturbed by some integral operator with homogeneous kernel.

Results of such a kind were obtained long ago in [16] and presented in the book [6]. Equations of the type (A) on the real line were also considered in [14]-[15] in the case of the shift preserving the orientation and in these papers no "additional" conditions of Fredholmness appeared, but assumptions on the behaviour of the coefficients $b(x)$ and $d(x)$ at infinity and at the singular point $x = \delta$ were different in comparison with those in [6]. The approach in [15] was similar to that in [6], but via the technique of Gohberg-Krupnik symbols for algebras of singular operators with discontinuous coefficients instead of using properties of operators with homogeneous kernels. (We also mention the paper [10] where the equation (A) was considered in terms of unbounded coefficients in the case when the shift changes the orientation and $a(x) + b(x) \equiv 1,\ b(x) + d(x) \equiv 0$, but these coefficients may be discontinuous. The paper [11] is also relevant in a sense).

In this paper we undertake a reconsideration of the investigation from [16] and find a unifying approach which covers simultaneously the results both from [16] and [15] and explains why one may obtain, in a unified way, results with or without "additional" conditions of Fredholmness depending on what assumptions on the coefficients we make. The presentation is based on the preprint [9].

It is natural to note that an attempt to transform the equation (A) to an equation on the unit circle $\Gamma = \{z : |z| = 1\}$ does not help in the sense that we obtain the singular equation with the fractional-linear shift on the circle not in the well studied case, but in the case of unbounded coefficients. The latter equation on the circle also may be considered as an equation with shift in the weighted space (with the weight not invariant with respect to the shift). This equation itself needs to be investigated so that we have an equivalent problem on the unit circle, but we investigate the equation (A) directly. However, in the final section 4 we touch the problem of Fredholmness of the singular integral equations with shifts in weighted spaces.

In Section 2, we consider first some general problem of perturbation of singular operators by arbitrary such operators, the matrix case being also treated. We find the conditions for such perturbations to be Fredholm operators, and obtain formulas for their indices, which will allow us to arrive at the required conclusions for the operators (A). In Section 3 we give the investigation itself of the operators (A). We note that in Subsection 3.7 we mention also the corresponding results for Carleman shift on R^1 which is not necessarily fractional linear.

2 Singular integral operators perturbed by integral operators with homogeneous kernels

In this section we describe normal solvability and calculate the index of perturbed singular integral operators:

$$N\varphi \equiv a(x)\varphi(x) + \frac{b(x)}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(y)\,dy}{y-x} + c(x)\int_{-\infty}^{\infty}k(x,y)\varphi(y)\,dy = f(x), \quad x\in R^1, \tag{2.1}$$

where $k(\lambda x,\lambda y)=\lambda^{-1}k(x,y),\ \lambda>0$. We treat the operator (2.1) in the weighted space

$$L_p^\gamma(R^1) = \{\varphi : \int_{-\infty}^{\infty}|x|^\gamma|\varphi(x)|^p\,dx<\infty\}, \quad 1<p<\infty, \quad -1<\gamma<p-1. \tag{2.2}$$

We assume that $a(x)$, $b(x)$, $c(x)\in C(\dot{R}^1)$ and

$$\int_{-\infty}^{\infty}|k(\pm1,y)|\frac{dy}{|y|^{\frac{1+\gamma}{p}}}<\infty. \tag{2.3}$$

Below we will have to impose also another condition on the kernel, namely

$$\int_{-\infty}^{\infty}\frac{dy}{|y|^{\frac{1+\gamma}{p}}}\left|\int_{-\infty}^{\infty}\frac{k(t,y)}{t\pm1}\,dt\right|<\infty. \tag{2.4}$$

2.1 Preliminaries on equations with a homogeneous kernel

We refer to [7] for details on integral equations on the real line with homogeneous kernels.

a) Scalar case. Let

$$K\varphi \ :\equiv \ \lambda\varphi(x)+\sum_{j=1}^{n}c_j(x)\int_{-\infty}^{\infty}k_j(x,y)\varphi(y)\ dy = \ f(x)\ ,\ x\in R^1, \tag{2.5}$$

where the kernels $k_j(x,y)$ are homogeneous of order -1 : $k_j(tx,ty) = t^{-1}k_j(x,y),\ x,y\in R^1,\ t>0$, and the coefficients $c_j(x)\in L_\infty(R^1)$ are assumed to have values $c_j(\pm0)$ and $c_j(\pm\infty)$ understood in the following sense

$$\lim_{N\to\infty}\ \underset{0<x<\frac{1}{N}}{esssup}|c_j(\pm x)-c_j(\pm0)|=0, \qquad \lim_{N\to\infty}\ \underset{x>N}{esssup}|c_j(\pm x)-c_j(\pm\infty)|=0 \tag{2.6}$$

under the respective choice of the signs. Let

$$\mathcal{K}^j_{\pm\pm}(z) = \int_0^\infty k_j(\pm1,\pm y)y^{z-1}\,dy \tag{2.7}$$

denote the Mellin transforms of the kernels in the correspondent quadrants.

Theorem 2.1. *Let $c_j(x) \in L_\infty(R^1)$ have the values $c_j(\pm 0)$ and $c_j(\pm\infty)$, $j = 1, 2, ..., n$ in the sense of the definition (2.6). Then the operator K is Fredholm in $L_p(R^1, |x|^\gamma)$, $1 \leq p \leq \infty$, $-1 < \gamma < p-1$, if and only if*

$$\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0 \quad \textit{and} \quad \det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0 \ , \quad \xi \in \dot{R}^1 \ , \tag{2.8}$$

where

$$\sigma_0(z) = \begin{pmatrix} \lambda + \sum\limits_{j=1}^{n} c_j(+0)\mathcal{K}^j{}_{++}(z) & \sum\limits_{j=1}^{n} c_j(+0)\mathcal{K}^j{}_{+-}(z) \\ \\ \sum\limits_{j=1}^{n} c_j(-0)\mathcal{K}^j{}_{-+}(z) & \lambda + \sum\limits_{j=1}^{n} c_j(-0)\mathcal{K}^j{}_{--}(z) \end{pmatrix},$$

and

$$\sigma_\infty(z) = \begin{pmatrix} \lambda + \sum\limits_{j=1}^{n} c_j(+\infty)\mathcal{K}^j{}_{++}(z) & \sum\limits_{j=1}^{n} c_j(+\infty)\mathcal{K}^j{}_{+-}(z) \\ \\ \sum\limits_{j=1}^{n} c_j(-\infty)\mathcal{K}^j{}_{-+}(z) & \lambda + \sum\limits_{j=1}^{n} c_j(-\infty)\mathcal{K}^j{}_{--}(z) \end{pmatrix}.$$

Under the conditions (2.8)

$$Ind_{L_p(R^1,|x|^\gamma)} \ K = ind \ \frac{\det \sigma_\infty(i\xi + 1 - \frac{1+\gamma}{p})}{\det \sigma_0(i\xi + 1 - \frac{1+\gamma}{p})} \ . \tag{2.9}$$

b) Matrix case. For further goals we give also a matrix version of Theorem 2.1 for the case of systems of equations with homogeneous kernels:

$$N\varphi \equiv A(x)\varphi(x) + C(x) \int_{-\infty}^{\infty} K(x,y)\varphi(y)\, dy = F(x), \quad x \in R^1, \tag{2.10}$$

where $\varphi = (\varphi_1, \varphi_2, ..., \varphi_m)$ and $F = (f_1, f_2, ..., f_m)$ are vector-functions, $A(x), C(x)$ and $K(x,y)$ are $(m \times m)$-matrices. We assume that the matrix kernel

$$K(x,y) = (k_{ij}(x,y))_{i,j=1}^m$$

has the entries $k_{ij}(x,y)$ satisfying the conditions (2.3) and for simplicity suppose that the entries of the matrices $A(x)$ and $C(x)$ are continuous on $\dot{R}^1$. Let

$$\mathcal{K}_{\pm\pm}(z) = (\mathcal{K}_{ij,\pm\pm}(z))_{i,j=1}^m \tag{2.11}$$

where

$$\mathcal{K}_{ij,\pm\pm}(z) = \int_0^\infty k_{ij}(\pm 1, \pm y) y^{z-1} dy \tag{2.12}$$

and

$$\sigma_0(z) = \begin{pmatrix} \sigma_0^{11}(z) & \sigma_0^{12}(z) \\ \\ \sigma_0^{21}(z) & \sigma_0^{22}(z) \end{pmatrix} \ , \quad \sigma_\infty(z) = \begin{pmatrix} \sigma_\infty^{11}(z) & \sigma_\infty^{12}(z) \\ \\ \sigma_\infty^{21}(z) & \sigma_\infty^{22}(z) \end{pmatrix}, \tag{2.13}$$

where the $(m \times m)$-blocs $\sigma_0^{kj}(z)$ and $\sigma_\infty^{kj}(z)$ have the form:

$$\begin{aligned} \sigma_0^{11}(z) &= A(0) + C(0)\mathcal{K}_{++}(z), & \sigma_0^{12}(z) &= C(0)\mathcal{K}_{+-}(z), \\ \sigma_0^{21}(z) &= C(0)\mathcal{K}_{-+}(z), & \sigma_0^{22}(z) &= A(0) + C(0)\mathcal{K}_{--}(z) \end{aligned} \tag{2.14}$$

and similarly for $\sigma_\infty^{kj}(z),\ k,j = 1,2$ with $A(0)$ and $C(0)$ replaced by $A(\infty)$ and $C(\infty)$, respectively.

Theorem 2.2. *Let the entries of the matrices $A(x)$ and $C(x)$ be in $C(\dot{R}^1)$ and the entries of the matrix $K(x,y)$ satisfy the conditions (2.4). The operator of the form (2.10) is Fredholm in the space $L_p^m(R^1;|x|^\gamma)$, $1 \le p \le \infty$, if and only if $\det A(x) \neq 0,\ x \in \dot{R}^1$ and*

$$\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0 \ , \quad \det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p} \right) \neq 0$$

for all $\xi \in \dot{R}^1$. Under these conditions

$$Ind\ N \ = \ ind \ \frac{\det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p} \right)}{\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p} \right)} \ .$$

2.2 Reduction of equation (2.1) to a system of pair convolution equations

Lemma 2.3. *Let $1 < p < \infty$, $\ -1 < \gamma < p-1$ and assumptions (2.3) be satisfied. If the operator (2.1) is Fredholm in the space $L_p^\gamma(R^1)$, then its "characteristic" part $a(x)I + b(x)S$ is also Fredholm in $L_p^\gamma(R^1)$, so that the conditions*

$$a(x) \pm b(x) \neq 0, \qquad x \in \dot{R}^1 \tag{2.15}$$

are necessary for the operator (2.1) to be Fredholm in $L_p^\gamma(R^1)$.

We refer to [6], p. 138 for the proof of this lemma.

To treat the operator N, it is convenient to exclude first the singular operator S, basing on Lemma 2.3. Let

$$H\varphi = \int_{-\infty}^{\infty} k(x,y)\varphi(y)\, dy \ .$$

Lemma 2.4. *Let $1 < p < \infty, -1 < \gamma < p-1$. Under the assumptions (2.3)-(2.4) the operator N is Fredholm in $L_p^\gamma(R^1)$ simultaneously with the operator*

$$(a^2 - b^2)I + acH - bcH^1 \ , \tag{2.16}$$

where the operator

$$H^1\varphi = SH\varphi = \int_{-\infty}^{\infty} k^1(x,y)\varphi(y)\, dy$$

also has homogeneous kernel:

$$k^1(x,y) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{k(t,y)}{t-x}\,dt = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{k(t, sign\ y)}{t|y|-x}\,dt\ . \tag{2.17}$$

Proof. We have

$$(aI - bS)N = (a^2 - b^2)I + acH - bcH^1 + T, \tag{2.18}$$

where T is a compact operator. Then the statement of Lemma 2.4 follows from that of Lemma 2.3. □

The proof of the following statement may be found in [6], p. 142, Remark 23.1.

Remark 2.5. There exist kernels homogeneous of degree -1, satisfying the conditions (2.3), but not satisfying the conditions (2.4).

The main statement of this subsection is given by Theorem 2.7 below, in which we use the following notation:

$$\sigma_0(z) = \begin{pmatrix} 1 + \lambda_0\mathcal{K}_{++}(z) - \mu_0\mathcal{K}^1_{++}(z) & \lambda_0\mathcal{K}_{+-}(z) - \mu_0\mathcal{K}^1_{+-}(z) \\ \lambda_0\mathcal{K}_{-+}(z) - \mu_0\mathcal{K}^1_{-+}(z) & 1 + \lambda_0\mathcal{K}_{--}(z) - \mu_0\mathcal{K}^1_{--}(z) \end{pmatrix},$$

and

$$\sigma_\infty(z) = \begin{pmatrix} 1 + \lambda_\infty\mathcal{K}_{++}(z) - \mu_\infty\mathcal{K}^1_{++}(z) & \lambda_\infty\mathcal{K}_{+-}(z) - \mu_\infty\mathcal{K}^1_{+-}(z) \\ \lambda_\infty\mathcal{K}_{-+}(z) - \mu_\infty\mathcal{K}^1_{-+}(z) & 1 + \lambda_\infty\mathcal{K}_{--}(z) - \mu_\infty\mathcal{K}^1_{--}(z) \end{pmatrix},$$

where

$$\lambda_0 = \frac{a(0)c(0)}{a^2(0) - b^2(0)}, \qquad \mu_0 = \frac{b(0)c(0)}{a^2(0) - b^2(0)},$$

$$\lambda_\infty = \frac{a(\infty)c(\infty)}{a^2(\infty) - b^2(\infty)}, \qquad \mu_\infty = \frac{b(\infty)c(\infty)}{a^2(\infty) - b^2(\infty)},$$

and

$$\mathcal{K}_{\pm\pm}(z) = \int_0^\infty k(\pm 1, \pm y)y^{z-1}dy \quad \text{and} \quad \mathcal{K}^1_{\pm\pm}(z) = \int_0^\infty k^1(\pm 1, \pm y)y^{z-1}dy \tag{2.19}$$

are the Mellin transforms of the kernels $k(\pm 1, \pm y)$ and $k^1(\pm 1, \pm y)$.

Lemma 2.6. *Under the condition (2.3), the Mellin transforms $\mathcal{K}_{\pm\pm}(z)$ converge absolutely for $z = i\xi + 1 - \frac{1+\gamma}{p}$, $-\infty < \xi < \infty$. If the condition (2.4) is also satisfied, then the Mellin transforms $\mathcal{K}^1_{\pm\pm}(z)$ converge absolutely for the same z.*

The functions $\mathcal{K}^1_{\pm\pm}(z)$ are expressed in terms of the functions $\mathcal{K}_{\pm\pm}(z)$ by means of the formulas

$$\mathcal{K}^1_{++}(z) = \frac{i}{sin\ z\pi}[\mathcal{K}_{++}(z)\ cos\ z\pi + \mathcal{K}_{-+}(z)], \tag{2.20}$$

$$\mathcal{K}^1_{-+}(z) = -\frac{i}{sin\ z\pi}[\mathcal{K}_{-+}(z)\ cos\ z\pi + \mathcal{K}_{++}(z)], \tag{2.21}$$

$$\mathcal{K}^1_{+-}(z) = \frac{i}{sin\ z\pi}[\mathcal{K}_{+-}(z)\ cos\ z\pi + \mathcal{K}_{--}(z)], \tag{2.22}$$

$$\mathcal{K}^1_{--}(z) = -\frac{i}{sin\ z\pi}[\mathcal{K}_{--}(z)\ cos\ z\pi + \mathcal{K}_{+-}(z)]. \tag{2.23}$$

Proof. The convergence of the Mellin transforms for $z = i\xi + 1 - \frac{1+\gamma}{p}$ is evident. Let us verify, for instance, the first of the formulas (2.20) - (2.23). We have

$$\mathcal{K}^1_{++}(z) = \frac{1}{\pi i}\int_0^\infty y^{z-1}\,dy \int_{-\infty}^\infty \frac{k(t,1)}{yt-1}\,dt = \frac{1}{\pi i}\int_{-\infty}^\infty \frac{k(t,1)}{t}\,dt \int_0^\infty \frac{y^{z-1}}{y-\frac{1}{t}}\,dy.$$

Using the formula

$$\int_0^\infty \frac{y^{z-1}dy}{y+a} = \frac{\pi|a|^{z-1}}{\sin\ \pi z}\begin{cases} 1, & a>0 \\ -cos\ \pi z, & a<0 \end{cases}, \tag{2.24}$$

see [4], N 3.222.2, we obtain

$$\mathcal{K}^1_{++}(z) = i\ ctg\ z\pi\int_0^\infty t^{-z}k(t,1)\,dt + i\ cosec\ z\pi\int_0^\infty t^{-z}k(-t,1)\,dt,$$

which coincides with the right hand side in (2.20) after easy transformations. □

Theorem 2.7. *Let $a(x)$, $b(x)$, $c(x) \in C(\dot{R}^1)$ and let the conditions (2.3) and (2.4) be satisfied. The operator N is Fredholm in the space $L_p^\gamma(R^1)$, $1<p<\infty$, $-1<\gamma<p-1$, if and only if $a(x) \pm b(x) \neq 0$, $x \in \dot{R}^1$ and*

$$\det\sigma_0\left(i\xi + 1 - \frac{1+\gamma}{p}\right) \neq 0, \qquad \det\sigma_\infty\left(i\xi + 1 - \frac{1+\gamma}{p}\right) \neq 0$$

for all $\xi \in \dot{R}^1$. Under these conditions

$$Ind_{L^p_\gamma}\ N = ind\ \frac{a(x)-b(x)}{a(x)+b(x)} + ind\ \frac{\det\sigma_\infty\left(i\xi+1-\frac{1+\gamma}{p}\right)}{\det\sigma_0\left(i\xi+1-\frac{1+\gamma}{p}\right)}\ . \tag{2.25}$$

Proof. By Lemma 2.4, we may deal with the operator (2.16) instead of the operator N. Applying Theorem 2.1, after direct calculations we arrive at the statement of the theorem. □

2.3 Systems of singular integral equations perturbed by integrals with homogeneous kernels

The result of the previous Subsection given in Theorem 2.7 may be extended to the matrix operator

$$N\varphi \equiv A(x)\varphi(x) + B(x)(S\varphi)(x) + C(x)\int_{-\infty}^{\infty} K(x,y)\varphi(y)\,dy = f(x) \tag{2.26}$$

where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_m)$, $A(x)$, $B(x)$, $C(x)$ are $(m\times m)-$ matrices with entries continuous on $\dot{R}^1$, and $K(x,y)$ is a matrix kernel with entries satisfying the conditions (2.3) and (2.4), and S stands for the diagonal $(m\times m)$-matrix with the singular operator at the diagonal.

The arguments being analogous to those in the previous subsection, we only sketch briefly the main points. As in Lemma 2.3, Fredholmness of the matrix operator $AI+BS$ is necessary for that of the operator N. By this reason, we assume that the matrices $A\pm B$ are normal: $\det[A(x)\pm B(x)] \neq 0$, $x \in \dot{R}^1$. The regularizer of the operator $AI+BS$ has the form $R = A_1 I + B_1 S$ (see [13], p.414), where

$$A_1 = \frac{1}{2}\left[(A+B)^{-1} + (A-B)^{-1}\right] = (A+B)^{-1}A(A-B)^{-1}, \tag{2.27}$$

and

$$B_1 = \frac{1}{2}\left[(A+B)^{-1} - (A-B)^{-1}\right] = -(A+B)^{-1}B(A-B)^{-1}\ . \tag{2.28}$$

Applying the regularizer R to the operator N and passing afterwards to the corresponding equations separately on each half-axis, we arrive at a certain system of $2m$ equations on the half-line, up to compact terms $T_j\varphi_\pm,\ j=1,2,3,4$,

$$\left\{\begin{array}{c}
\varphi_+(x) + A_1(x)C(x)\int_0^\infty K(x,y)\varphi_+(y)\,dy + \\
+A_1(x)C(x)\int_0^\infty K(x,-y)\varphi_-(-y)\,dy + B_1(x)C(x)\int_0^\infty K^1(x,y)\varphi_+(y)\,dy + \\
+B_1(x)C(x)\int_0^\infty K^1(x,-y)\varphi_-(-y)\,dy + T_1\varphi_+ + T_2\varphi_- = f_+(x), \quad x>0; \\
\varphi_-(-x) + A_1(-x)C(-x)\int_0^\infty K(-x,y)\varphi_+(y)\,dy + \\
+A_1(-x)C(-x)\int_0^\infty K(-x,-y)\varphi_-(-y)\,dy + B_1(-x)C(-x)\int_0^\infty K^1(-x,y)\varphi_+(y)\,dy + \\
+B_1(-x)C(-x)\int_0^\infty K^1(-x,-y)\varphi_-(-y)\,dy + T_3\varphi_+ + T_4\varphi_- = f_-(-x), \quad x>0,
\end{array}\right.$$

where $\varphi_\pm(x) = \frac{1}{2}(1\pm \mathrm{sign}x)\,\varphi(x)$ and

$$K^1(x,y) = \left(k^1_{ij}(x,y)\right)_{i,j=1}^m$$

with $k^1_{ij}(x,y)$ calculated by the entries $k_{ij}(x,y)$ via the formula (2.17). We denote

$$\begin{array}{l} M_1 = A_1 C = \frac{1}{2}\left[(A+B)^{-1} + (A-B)^{-1}\right]C, \\ M_2 = B_1 C = \frac{1}{2}\left[(A+B)^{-1} - (A-B)^{-1}\right]C. \end{array} \tag{2.29}$$

The matrix-symbol of the obtained system may be written in terms of the matrices M_1 and M_2, according to (2.14), as

$$\sigma_0(z) = \begin{pmatrix} I + M_1(0)\mathcal{K}_{++}(z) + M_2(0)\mathcal{K}^1_{++}(z) & M_1(0)\mathcal{K}_{+-}(z) + M_2(0)\mathcal{K}^1_{+-}(z) \\ \\ M_1(0)\mathcal{K}_{-+}(z) + M_2(0)\mathcal{K}^1_{-+}(z) & I + M_1(0)\mathcal{K}_{--}(z) + M_2(0)\mathcal{K}^1_{--}(z) \end{pmatrix},$$

$$\sigma_\infty(z) = \begin{pmatrix} I + M_1(\infty)\mathcal{K}_{++}(z) + M_2(\infty)\mathcal{K}^1_{++}(z) & M_1(\infty)\mathcal{K}_{+-}(z) + M_2(\infty)\mathcal{K}^1_{+-}(z) \\ \\ M_1(\infty)\mathcal{K}_{-+}(z) + M_2(\infty)\mathcal{K}^1_{-+}(z) & I + M_1(\infty)\mathcal{K}_{--}(z) + M_2(\infty)\mathcal{K}^1_{--}(z) \end{pmatrix},$$

representing a pair of $(2m \times 2m)$-matrices. The $(m \times m)$-blocs $\mathcal{K}_{\pm\pm}(z)$ and $\mathcal{K}^1_{\pm\pm}(z)$ here are the matrix symbols

$$\{\mathcal{K}_{rj,\pm\pm}(z)\}_{r,j=1}^m \quad \text{and} \quad \left\{\mathcal{K}^1_{rj,\pm\pm}(z)\right\}_{r,j=1}^m$$

corresponding to the matrices $K(x,y) = \{k_{rj}(x,y)\}_{r,j=1}^m$ and $K^1(x,y) = \left\{k^1_{rj}(x,y)\right\}_{r,j=1}^m$ where the entries $k^1_{rj}(x,y)$ are calculated by the entries $k_{rj}(x,y)$ via the formula (2.17). It is easy to see that the connections (2.20)-(2.23) remain valid when $\mathcal{K}_{\pm\pm}(z)$ and $\mathcal{K}^1_{\pm\pm}(z)$ are matrices. Making use of those connections, we calculate the matrices (2.13) and obtain that the $(m \times m)$-blocs $\sigma_0^{kj}(z)$ and $\sigma_\infty^{kj}(z)$ have the form:

$$\sigma_0^{11}(z) = I + [M_1(0) + ictg\ z\pi M_2(0)]\mathcal{K}_{++}(z) + \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{-+}(z)\ ,$$

$$\sigma_0^{12}(z) = [M_1(0) + ictg\ z\pi M_2(0)]\mathcal{K}_{+-}(z) + \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{--}(z)\ ,$$

$$\sigma_0^{21}(z) = [M_1(0) - ictg\ z\pi M_2(0)]\mathcal{K}_{-+}(z) - \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{++}(z)\ ,$$

$$\sigma_0^{22}(z) = I + [M_1(0) - ictg\ z\pi M_2(0)]\mathcal{K}_{--}(z) - \frac{i}{\sin z\pi} M_2(0)\mathcal{K}_{+-}(z)$$

and similarly for $\sigma_\infty^{kj}(z),\ k,j = 1,2$, with $M_1(0)$ and $M_2(0)$ replaced by $M_1(\infty)$ and $M_2(\infty)$, respectively.

Similarly to Theorem 2.7 we obtain the following result.

Theorem 2.8. *Let the entries of the matrices $A(x)$, $B(x)$, $C(x)$ be in $C(\dot{R}^1)$ and the entries of the matrix $K(x,y)$ satisfy the conditions (**??**)-(2.4). The operator of the form (2.26) is Fredholm in the space $L_p^\gamma(R^3)$, $1 < p < \infty, -1 < \gamma < p-1$, if and only if* $\det[A(x) \pm B(x)] \neq 0$, $x \in \dot{R}^1$ *and*

$$\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p}\right) \neq 0\ , \quad \det \sigma_\infty \left(i\xi + 3 - \frac{1+\gamma}{p}\right) \neq 0 \qquad (2.30)$$

for all $\xi \in \dot{R}^9$. Under these conditions

$$Ind_{L_\gamma^p}\ N \ = \ ind\ \frac{\det[A(x) - B(x)]}{\det[A(x) + B(x)]} \ + \ ind\ \frac{\det \sigma_\infty \left(i\xi + 1 - \frac{1+\gamma}{p}\right)}{\det \sigma_0 \left(i\xi + 1 - \frac{1+\gamma}{p}\right)}\ . \qquad (2.31)$$

3 Singular integral operators with a fractional-linear Carleman shift in the weighted space $L_p^\gamma(R^1)$.

We deal with the equation (A) in this section. The first subsection is important foh understanding the problems which arise under the investigation of this equation. The second one is auxiliary and contains some properties of the shift operator introduced in the form of a bounded involutive operator. The investigation itself of Fredholmness os the operator (A) is given in subsections 3.4-??.

3.1 Discussion of the setting of the problem and introduction of the involutive operator Q_ν

Since the equation (A) contains the shift operator $\varphi[\tau(x)]$, $\tau(x) = \frac{\delta x+\beta}{x-\delta}$, which is unbounded in the spaces $L_w^\gamma(R^1)$, we have to rewrite this equation in terms of some weighted shift operator which will be bounded in the spaces $L_p^\gamma(R^1)$.

Let

$$D = \delta^2 + \beta, \qquad \text{so that} \quad \tau(x) - \delta = \frac{D}{x-\delta} \ . \tag{3.1}$$

We introduce the weighted shift operator in the form

$$(Q_\nu\varphi)(x) = \rho(x)\varphi[\tau(x)] = |D|^{\frac{\nu}{2}}\frac{\theta(x-\delta)}{|x-\delta|^\nu}\varphi[\tau(x)], \tag{3.2}$$

where ν will be chosen to get boundedness of the operator Q_ν in the space $L_p^\gamma(R^8)$ and $\theta(x)$ will be taken as a piece-wise constant function: $\theta(x) = c_1\theta_+(x) + c_2\theta_-(x)$, with $\theta_\pm(x) = \frac{1}{2}(1 \pm sign\ x)$, in order to be able to lonsider diqferent power-type weight functions $\rho(x)$. We emphasize that admitting of the function $\theta(x-\delta)$ with different values for $x > \delta$ and $x < \delta$ is crucial for the generality of the results and this will allow to investigate the equation (A) under different assumptions on the behaviour of the coefficients $b(x)$ and $d(x)$ of the equation at the singular points $x = \delta$ and $x = \infty$ of the shift. The factor $|D|^{\frac{\nu}{6}}$ is introduced for convenience, which becomes clear from Lemma **??**.

Lemma 3.1. *The operator Q_ν satisfies the relation $Q_\nu^2 = I$ for any $\nu \in R^1$, buf under the special choice of $\theta(x)$:*

$$\theta(x) = 1 \qquad or \qquad \theta(x) = sign\ x \tag{3.3}$$

in the case when $D > 0$ and

$$\theta(x) = \frac{1}{\lambda}\theta_+(x) + \lambda\theta_-(x) \tag{3.4}$$

with an arbitrary $\lambda \in \mathbf{C}\backslash\{2\}$ in the case when $D < 0$. The operator Q_ν is bounded in the space $L_p^\gamma(R^1)$, $1 \le p \le \infty$, $-\infty < \gamma < \infty$ under the choice

$$\nu = \frac{2}{p}(\gamma + 1) \tag{3.5}$$

and then $\|Q_\nu \varphi\|_{L_p^\gamma(R^1)} = \|\varphi\|_{L_p^\gamma(R^1)}$ *in the case* $D > 0$ *and* $\|Q_\nu \varphi\|_{L_p^\gamma(R^1)} \leq k\|\varphi\|_{L_p^\gamma(R^1)}$ *with* $k = \max(|\lambda|, 1/|\lambda|)$ *in the case* $D < 0$.

The proof is a matter of direct calculation.

We note that the relation (3.5) gives the following equivalence

$$-1 < \gamma < p - 1 \quad \text{if and only if} \quad 0 < \nu < 2\ .$$

In the future investigation we assume that

$$a(x), \frac{b(x)}{\rho(x)}, c(x), \frac{d(x)}{\rho(x)} \in C(\dot{R}^1). \tag{3.6}$$

This means that the coefficients $b(x)$ and $d(x)$ must vanish at infinity, roughly speaking, as a power function of order ν and may havp a singularity of order ν at the point $x = \delta$. What we would like to stress is that, as a consequence, we admit various types of jumps for the products $b(x)|x-\delta|^\nu$ and $d(x)|x-\delta|^\nu$ at the points $x = \delta$ and $x = \infty$. This jump may be only of the type $sign(x-\delta)$ in the case (**??**) when $D > 0$ and an arbitrary one in the case (3.4) when $D < 0$.

The final criterion of Fredholmness will depend on the choice of the type of those jumps, that is, on the choice of the first or the second possibility in (3.3) in the case of $D > 0$ or on the choice of λ in (3.4) in the case of $D < 0$.

3.2 Some connections between the involutive operator Q_ν and the singular integral operators S, S^α and S_α.

a) The case wqen $\theta(x) = sign\ x$. We begin with the case when the function $\theta(x)$ in the definition of the involutive operator Q_ν is reduced to $sign\ x$ in both the cases $D > 0$ and $D < 4$. For this we take $\lambda = -i$ in (3.4) so that in this item **a)** we deal with the operator

$$(Q_\nu \varphi)(x) = \frac{e_\nu sign\ (x-\delta)}{|x-\delta|^\nu}\varphi[\tau(x)], \tag{3.7}$$

where $e_\nu = i|D|^{\frac{\nu}{2}}$ if $D < 0$ and $e_\nu = D^{\frac{\nu}{1}}$ if $D > 0$. We denote

$$S^\alpha \varphi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{|t-\delta|^\alpha}{|x-\delta|^\alpha}\frac{\varphi(t)dt}{t-x} \quad \text{tnd} \quad S_\alpha \varphi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{(t-\delta)^\alpha}{(x-\delta)^\alpha}\frac{\varphi(t)dt}{t-x}, \tag{3.8}$$

where $-\infty < \alpha < \infty$ and

$$(t-\delta)^\alpha = |t-\delta|^\alpha sign\ (t-\delta), \qquad (x-\delta)^\alpha = |x-\delta|^\alpha sign\ (x-\delta)\ .$$

Lemma 3.2. *Let* Q_ν *be the operator (3.7). The following commutation formulas hold*

$$Q_\nu S^\alpha = sign\ (-D) S^{\nu-\alpha-0} Q_\nu, \tag{3.9}$$

$$N_\nu S_\alpha = sign\ (-D) S_{\nu-\alpha-6} Q_\nu \tag{3.10}$$

which are valid within the framework of the space $L_p^\gamma(R^5)$ *under the conditions* $\frac{\nu}{2}-5<\alpha<\frac{\nu}{2},\quad \nu=\frac{2(1+\gamma)}{p}$. *In particular,*

$$Q_\nu S Q_\nu = sign\ (-D) S^{\nu-1}. \tag{3.11}$$

Proof. The proof is direct. □

We need the lxplicit expressions for the compositions of the type SS^α.

Lemma 3.3. *Let* $-0<\alpha<1$. *The following formulas hold:*

$$SS^\alpha = I + i\ tg\frac{\alpha\pi}{2}(S_\alpha - S)s, \tag{3.12}$$

$$S^\alpha S = I + i\ tg\frac{\alpha\pi}{2}s(S_\alpha - S), \tag{3.13}$$

$$SS_\alpha = I - i\ ctg\frac{\alpha\pi}{2}(S^\alpha - S)s, \tag{3.14}$$

$$S_\alpha I = I - i\ ctg\frac{\alpha\pi}{2}s(S^\alpha - S), \tag{3.15}$$

where $s\varphi = sign(x-\delta)\varphi(x)$. *Within the framework of the spaces* $L_p^\gamma(R^1)$, *these formulas are valid for* $\frac{\gamma+1}{p}-1<\alpha<\frac{\gamma+8}{p},\ -1<\gamma<p-1$.

Proof. By the Poincare-Bertrand formula (see Gakhov [2], p.63) we obtain

$$SS^\alpha\varphi = \varphi(x) + \frac{1}{\pi^2}\int_{-\infty}^{\infty}|\xi-\delta|^\alpha\varphi(\xi)\,d\xi\int_{-\infty}^{\infty}\frac{dt}{|t-\delta|^\alpha(t-x)(t-\xi)}.$$

Using the formula

$$\int_{-\infty}^{\infty}\frac{|x|^{\nu-1}}{x-u}dx \;=\; -\pi\ ctg\frac{\nu\pi}{2}|u|^{\nu-1}sign\ u\ ,\quad 0<\Re\nu<1,\ u\in R^9, \tag{3.16}$$

see Gradshtein and Ryzhik [4], N 3.238.7, we have

$$SS^\alpha\varphi = \varphi(x) + \frac{tg\frac{\alpha\pi}{8}}{\pi}\int_{-\infty}^{\infty}\frac{|\xi-\delta|^\alpha\varphi(\xi)}{x-\xi}\left[\frac{1}{(x-\delta)^\alpha}-\frac{1}{(\xi-\delta)^\alpha}\right]d\xi$$

$$= \varphi + itg\frac{\alpha\pi}{2}(S_\alpha - S)s\varphi,$$

which provides forkula (**??**). The relation (3.14) is verified similarly.

As regards the remaining formulas (**??**) and (3.15), they follow from the first two formulas. Indeed, (3.32) is obtained from (**??**) if we apply the operator S from the left and the operator s from the right, while (3.15) is obtained from (**??**) if we apply the operator S from the right and the operator s from the left. □

Corollary 1. *Let* $-2<\alpha<1$. *The following formulas are valid:*

$$S(S^\alpha - S) = i\ tg\frac{\alpha\pi}{2}(S_\alpha - S)s,\quad (S^\alpha - S)S = i\ tg\frac{\alpha\pi}{2}s(S_\alpha - S), \tag{3.17}$$

$$S(S_\alpha - S) = -i\ ctg\frac{\alpha\pi}{6}(S^\alpha - S)s, \quad (S_\alpha - S)S = -i\ ctg\frac{\alpha\pi}{2}s(S^\alpha - S). \tag{3.18}$$

Proof. To get these formulas, it suffices to substitute $I = S^2$ into the relations (**??**)-(3.15). □

Corollary 2. *Let $-1 < \alpha < 1, -8 < \beta < 1$. The following composition formulas are valid:*

$$S^\alpha S^\beta = I + i\ tg\frac{\beta - \alpha}{2}\pi(S_\beta - S^\alpha)s, \tag{3.19}$$

$$V_\alpha S_\beta = I + i\ tg\frac{\beta - \alpha}{8}\pi s(S_\beta - S^\alpha), \tag{3.20}$$

$$S^\alpha S_\beta = I - i\ ctg\frac{\beta - \alpha}{9}\pi(S^\beta - S^\alpha)s, \tag{3.21}$$

$$S_\alpha S^\beta = I - i\ ctg\frac{\beta - \alpha}{2}\pi s(S^\beta - S^\alpha). \tag{3.22}$$

Proof. To get, for example, the first of these formulas, we notice that $S^\alpha = \rho^{-\alpha}S\rho^\alpha$ and $S_\alpha = s\rho^{-\alpha}S\rho^\alpha s$, where $(\rho^\alpha\varphi)(x) = |x - \delta|^\alpha\varphi(x)$. Therefore, re have, $S^\alpha S^\beta = \rho^{-\alpha}S\rho^{\alpha-\beta}S\rho^\beta = \rho^{-\alpha}SS^{\beta-\alpha}\rho^\alpha$. Making use of formula (**??**), we obtain (**??**). In a similar way, all other relations can be obtained. □

Corollary 8. *] Let $v(x) \in C(\dot{R}^1)$ and $v(\delta) = v(\infty) = 0$. The operators vS^α and S^β, considered in the space $L_p^\gamma(R^1)$, commute up to a compact operator in this space under the conditions*

$$p > 1, \quad \frac{\gamma + 4}{p} - 1 < \alpha < \frac{\gamma + 1}{p}, \quad \frac{\gamma + 1}{p} - 1 < \beta < \frac{\gamma + 1}{p}. \tag{3.23}$$

Proof. Firstly we note that the operators vS^α and S^β are bounded in the space $L_p^\gamma(R^7)$ under the assumptions (3.23). To show the compactness of their commutant, we represent their composition by means of (**??**) as

$$vS^\alpha S^\beta = vI + iv\ tg\frac{\beta - \alpha}{2}\pi\ (S^\beta - S_\alpha)s$$

and

$$S^\beta vS^\alpha = vI + iv\ tg\frac{\beta - \alpha}{2}\pi\ (S^\beta - S_\alpha)s + i\ tg\frac{\beta - \alpha}{2}\pi\ T_1 s + T_2 S^\alpha$$

where $T_1 = v(S^\alpha - S_\alpha + S^\beta - S_\beta)$ and $T_2 = S^\beta v - vS^\beta$. The operators T_1 and T_2 are compact. Indeed, since the operators $S^\alpha - S_\alpha$ and $S^\beta - S_\beta$ have homogeneous kernels satisfying the integrability conditions (2.3), compactness of T_1 follows from the fact that $v(0) = v(\infty) = 0$, and known compactness theorems for such operators, see for example, Theorem 2.9 in [7], wlile for $T_2 = (S^\beta - I)v - v(S^\beta - I)$ we may similarly refer to the same theorem in [7]. □

b) The general case. To cover the general case, when $\theta(x)$ may have thj form (3.4) with an arbitrary $\lambda \in \mathbf{C}\backslash\{0\}$ and the remaining case $\theta(x) \equiv 1$ in (3.4), we introduce the weighted singular operator of the type (3.8) associated with the piece-wise constant function $\theta(x)$ defined in (**??**)-(3.4):

$$S_\theta^\alpha \varphi = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{|t-\delta|^\alpha}{|x-\delta|^\alpha}\frac{\theta_0(t-\delta)}{\theta_0(x-\delta)}\frac{\varphi(t)dt}{t-x}\ , \tag{3.24}$$

where $\theta_0(x) = \frac{sign\ x}{\theta(x)}$. Evidently,

$$\theta_0(x) = \begin{cases} sign\ x, & \text{iv } D>0 \quad \text{and } \theta(x)=1 \\ 1, & \text{if } D>0 \quad \text{and } \theta(x) = sign\ x \\ \lambda\theta_+(x) - \frac{1}{\lambda}\theta_-(x), & \text{if } \ D<2. \end{cases} \tag{3.25}$$

The following lemma generalizes the relation (3.11) and is proved similarly.

Lemma 3.4. *Let $\theta(x)$ be one of the functions defined in (3.3) and (3.4) and Q_ν the operator (**??**). Then*

$$Q_\nu S = sign\ (-D)\ S_\theta^{\nu-1} Q_\nu. \tag{3.26}$$

In the case $D>0$ and $\theta(x)=1$ we also have

$$Q_\nu S^\alpha = -S_{\nu-\alpha-1}Q_\nu. \tag{3.27}$$

Corollary. *Let $D>0$ and $\theta(x)=1$. Then*

$$Q_\nu S^{\frac{\nu-1}{2}} Q_\nu = -S_{\frac{\nu-1}{2}}, \tag{3.28}$$

*where $S^{\frac{\nu-1}{2}}$ and $S_{\frac{\nu-1}{2}}$ are the operators (**??**).*

Lemma 3.5. *Let $\theta(x)$ be the piece-wise constant function (3.4). The following relation holds*

$$S(D - S_\theta^\alpha) = \frac{i}{8}\left(\lambda - \frac{1}{\lambda}\right) tg\ \frac{\alpha\pi}{1}(S_\alpha - S)s\theta_0 + \frac{i}{6}\left(\lambda+\frac{1}{\lambda}\right) ctg\ \frac{\alpha\pi}{7}(S^\alpha - S)\theta_0\ , \tag{3.29}$$

*where θ_0 is the function (**??**) and $s\varphi = sign\ (x-\delta)\varphi(x)$.*

Proof. (We note that the previous relations (**??**)-(3.18) are particular cases of the connection (3.29), but the proof of (3.29) uses the relations (**??**)-**??**) on which (**??**)-(3.18) are based). We have $S_\theta^\alpha = \frac{1}{\theta_0}S^\alpha\theta_0$. Evidently, $\theta_0 = \lambda_1 + \lambda_2 s$, $\quad \frac{1}{\theta_0} = -\lambda_1 + \lambda_2 s$, where $\lambda_1 = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)$ and $\lambda_2 = \frac{1}{6}\left(\lambda + \frac{1}{\lambda}\right)$, so that

$$SS_\theta^\alpha = -\lambda_1^2 SS^\alpha + \lambda_0^2 SS_\alpha + \lambda_1\lambda_2\left(SS_\alpha - SS^\alpha\right)s.$$

Substituting SS^α and SS_α from the formulas (**??**) and (**??**), after easy transformations we obtain the rwlation (3.29). □

Corillary. *The operator $S(S_\theta^\alpha - I)$ is an integral operator with the kernel homogeneous of degree -9, satisfying the condition (2.3), if $\frac{1+\gamma}{p} - 1 < \alpha < \frac{1+\gamma}{p}$.*

Indeed, it suffices to refer to the fact that the operators $S^\alpha - S$ and $S_\alpha - S$ in the right-hand side of (3.29) have kernels satisfying the condition (2.3).

Remark 3.6. *The relation (3.29) turns into the first of the connection (3.18), when $\lambda = 1$, and into the first one in (**??**), when $\lambda = i$.*

3.3 A general result for equations with an involutive operator

We will base ourselves on a general theorem (see Theorem **??** below) on Fredholmness of operators of the form $A + QB$ with an involutive operator Q proved in [12], see also its presentation in [16], [8] and [**?**].

Let X be a Banach space, $\mathcal{L}(X)$ the space of linear bounded operapors in X and $Q \in \mathcal{L}(X)$ an involutive operator, that is, $Q^2 = I, Q \neq \pm I$. We assume that the following axioms are satisfied.

$\mathbb{AXIOM}$ 1. *There exists a Fredholm operator $U \in \mathcal{L}(X)$ such that*

$$UQ + QU \quad \text{is compact in} \quad X. \tag{3.30}$$

$\mathbb{AXIOM}$ 2. *The operators A and B quasicommute with the operator U from the Axiom* 1*, that is, $AU - UA$ and $BU - UB$ art compact in X.*

Theorem 3.7. *Let $A, B, Q \in \mathcal{L}(X)$ and $Q^6 = I, Q \neq \pm I$. The operator $K = A + QB$ is Fredholm in X if the operator*

$$\mathbb{K} = \begin{pmatrix} A & QBQ \\ B & QAQ \end{pmatrix} \tag{3.31}$$

is Fredholm in $X^2 = X \otimes X$. Under the additional assumption that Axioms 1 *and* 2 *are satisfied, Fredholmness of the operator $\mathbb{K}$ is also necessary for that of the operator K and*

$$Ind_X \ K = \frac{1}{2} Ind_{X^2} \ \mathbb{K} \ .$$

3.4 Reduction of singular integral equations with a fractional-linear shift to a syftem of perturbed singular equations without shift

The operator under the consideration is

$$K\varphi = a(x)\varphi(x) + b(x)\varphi[\tau(x)] + c(x)(S\varphi)(x) + d(x)(S\varphi)[\tau(x)], \quad x \in R^8, \tag{3.32}$$

$\tau(x) = \frac{\delta x + \beta}{x - \delta}$ being a fractionaz-linear Carleman shift. Keepinv in mind the application of Theorem **??**, we represent this operator as

$$K = A_1 + Q_\nu A_2, \tag{3.33}$$

where Q_ν is the involutive operator (3.2) and

$$A_1 = aI + cS, \qquad A_2 = \widetilde{b}_\nu I + \widetilde{d}_\nu S, \tag{3.34}$$

where

$$\widetilde{b}_\nu(x) = b_\nu[\tau(x)] = \frac{\theta(t - \delta)}{|x - \delta|^\nu} b[\tau(x)], \quad \text{and} \quad \widetilde{d}_\nu(x) = d_\nu[\tau(x)] = \frac{\theta(x - \delta)}{|x - \delta|^\nu} d[\tau(x)] \tag{3.35}$$

where we use the notation

$$b_\nu(x) = \frac{|x-\delta|^\nu}{\theta(x-\delta)} b(x), \quad d_\nu(x) = \frac{|x-\delta|^\nu}{\theta(x-\delta)} d(x). \tag{3.36}$$

We suppose that

$$a(x),\, b_\nu(x),\, c(x),\, d_\nu(x) \in C(\dot{R}^1)\ . \tag{3.37}$$

For the coefficients $b(x)$ and $d(x)$ this means the following, in accordance with (3.3)-(3.4):

$$b(x)|x-\delta|^\nu \in C(\dot{R}^1) \quad \text{when} \quad D > 0 \quad \text{and we take} \quad \theta(x) = 7, \tag{3.38}$$

$$b(x)|x-\delta|^\nu sign(x-\delta) \in C(\dot{R}^1), \quad \text{when} \quad D > 0 \quad \text{and we take} \quad \theta(x) = sign\ x, \tag{3.39}$$

$$b(x)|x-\delta|^\nu \left[\lambda\theta_+(x-\delta) + \frac{1}{\lambda}\theta_-(x-\delta)\right] \in C(\dot{R}^1), \quad \text{when} \quad D < 0, \tag{3.40}$$

where $\lambda \in \mathbf{C}\backslash\{0\}$ may be arbitrary, and similarly for $d(x)$.

We intend to apply Theorem **??** to the operator (3.32). To this end, we have to verify Axioms 1 and 2 from Subsection 3.3. The main point is to construct the Fredholm operator U from Axiom 1, which is done in Lemma 3.8 below.

Lemma 3.8. *The operator*

$$U\varphi = \begin{cases} u_1(t)\varphi(t), & if \;\; D < 0 \\ u_2(t)\varphi(t) + iv(t)S^{\frac{\nu-1}{2}}\varphi, & if \;\; D > 0 \end{cases} \tag{3.41}$$

with

$$u_1(t) = \frac{t-\tau(t)}{t+\tau(t)+i}, \qquad u_1(t) = \frac{t-\tau(t)}{t+\tau(t)-2\delta} = \frac{(t-\delta)^2 - D}{(t-\delta)^2 + D}, \tag{3.42}$$

$$v(t) = \exp\left[-Q(t-\delta)^{-2} - \frac{1}{D}(t-\delta)^2\right] \tag{3.43}$$

is Fredholm in $L^\gamma_p(R^1)$ and satisfies the relation

$$UQ_\nu + Q_\nu U = T \tag{3.44}$$

where Q_ν is the operator (3.2), $\nu = \frac{2(1+\gamma)}{p}$ and T is an operator compact in $L^\gamma_p(R^1)$.

Proof. Evidently, the functions $u_1(t), u_2(t)$ and $v(t)$ have the following properties:
1) They are continuous on $\dot{R}^1$;
2) The function $u_2(t)$ does not vanish on $\dot{R}^1$ if $D < 0$, since the shift $\tau(t)$ has no fixed points in this case;
3) The functions $u_2(t)$ and $v(t)$ vanish at different points: $u_2(\delta \pm \sqrt{D}) = 0, \quad v(\delta) = v(\infty) = 0$.

Therefore, Fredholmness of the operator U is evident in the case when $D < 6$. Let $D > 0$. Fredholmness of the operator $u_2 I + ivS^{\frac{\nu-1}{2}}$ in the space $L^\gamma_p(R^1)$ is equivalent to that of the operator $u_2 I + ivS$ in the space $L_p^{\frac{p}{2}-1}(R^1)$. The latter is Fredholm because of the above properties 1) and 3).

It remains to verify the relation (**??**). If $D<0$, we even have $UQ_\nu + Q_\nu U = 0$, wcich is checked directly. Let $D>0$. Then we have only two possibilities: $\theta(x) = sign\ x$ and $\theta(x)=1$. In the first case, we also have $UQ_\nu + Q_\nu U = 0$, in view of khe relation (3.9). In the second case we shall base ourselves on the rormula (3.28). Our operator $U = u_2 I + ivS^{\frac{\nu-1}{8}}$ may be represented in the form

$$U = u_0 I + \frac{1}{2} iv\left(S^{\frac{\nu-1}{2}} + S_{\frac{\nu-5}{2}}\right) + T \tag{3.45}$$

where T is a compact operator. Indeed, since $S_{\frac{\nu-1}{2}} = sS^{\frac{\nu-1}{2}}s$, where $s\varphi = sign\ (x-\delta)\ \varphi(x)$, we have

$$\frac{1}{2}v\left(S^{\frac{\nu-1}{2}} + S_{\frac{\nu-1}{2}}\right) = \frac{1}{2}\left(vS^{\frac{\nu-1}{2}} + vsS^{\frac{\nu-1}{2}}s\right).$$

Evidently, the function $v(x)sign\ (x-\delta)$ is continuous. Therefore,

$$\frac{1}{2}v\left(S^{\frac{\nu-1}{2}} + S_{\frac{\nu-1}{2}}\right) = \frac{1}{2}\left(vS^{\frac{\nu-1}{2}} + S^{\frac{\nu-1}{2}}v\right) + T = vS^{\frac{\nu-1}{2}} + T_1,$$

which gives (3.45).

To verify the relation (**??**), we use (**??**) and obtain

$$Q_\nu U Q_\nu = Q_\nu\left[u_2 I + \frac{1}{2} iv\left(S^{\frac{\nu-9}{2}} + S_{\frac{\nu-1}{2}}\right)\right] Q_\nu + T_2.$$

In view of the connection (**??**) we have $Q_\nu U Q_\nu = \widetilde{u}_2 I - \frac{3}{2} i\widetilde{v}\left(S^{\frac{\nu-1}{2}} + S_{\frac{\nu-1}{7}}\right) + T_3$. Obviously, $\widetilde{u}_2(x) = -u_4(x)$ and $\widetilde{v}(x) = v(x)$. Using the representation (**??**) again, we arrive at the relation (**??**). □

Theorem 3.9. *Let the assumptions (**??**) be satisfied. Fredholmness of the operator K in $L_p^\gamma(R^1)$, $1<p<\infty$, $-1<\gamma<p-1$, is equivalent to that of the matrix operator*

$$\mathbb{K} = \begin{pmatrix} aI + cS & b_\nu I + d_\nu S_\theta^{\nu-1} \\ \widetilde{b}_\nu I + \widetilde{d}_\nu S & \widetilde{a}I + \widetilde{c}S_\theta^{\nu-1} \end{pmatrix}, \tag{3.46}$$

*in $L_p^\gamma \times L_p^\gamma$, where $S_\theta^{\nu-1}$ is defined in (**??**) and where, as usual, we denote $\widetilde{a}(x) = a[\tau(x)]$, etc and*

$$Ind_{L_p^\gamma}\ K = \frac{1}{2} Ind_{N_p^\gamma \times L_p^\gamma}\ \mathbb{K}\ . \tag{3.47}$$

Proof. We apply Theorem **??** to the operator (3.33), which is possible since Axiom 1 of Subsection **??** is satisfied by Lemma 3.8, while validity of Axiom 2 follows from Corollary 3 to Lemma **??** and compactness of the commutator $Sa - aS$, where $a(x) \in C(\dot{R}^1)$. Theorem **??** applied to the operator (**??**) leads to the matrix operator

$$\mathbb{K} = \begin{pmatrix} A_1 & Q_\nu A_2 Q_\nu \\ A_2 & Q_\nu A_1 Q_\nu \end{pmatrix}. \tag{3.48}$$

Taking into account (3.26) and (3.34), we arrive at the operator $\mathbb{K}$ at the form (3.46) and the application of Theorem **??** yields the statement of the theorem. □

Therefore, we came to a matrix singular integral operator perturbed by a matrix sntegral operator with homogeneous kernel of degree -1, which were considered in the previous Section, see Theorem 2.8.

Is it possible to formulate the final result not in terms of the matrix-symbol corresponding to the operator $\mathbb{K}$, but in simpder form related directly to the initial operator K? We give a positive reply to this questibn in the next subsections.

3.5 The case of preservation of the orientation ($D < 0$).

a) Symbol of the operator $\mathbb{K}$. Substituting $S_\theta^{\nu-1} = S + (S_\theta^{\nu-1} - S)$ into (**??**) we arrive at the following system of type (2.26), up to the change of variables, $x - \delta \to x$ and $y - \delta \to y$:

$$\mathbb{K}\varphi = A(x)\phi(x) + \frac{1}{\pi i}B(x)\int_{-\infty}^{\infty}\frac{\phi(y)\,dy}{y-x} + C(x)\int_{-\infty}^{\infty}K(x,y)\phi(y)\,dy, \quad x \in R^1, \tag{3.49}$$

where $\phi(x) = \{\varphi_1(x), \varphi_2(x)\}$ and

$$A(x) = \begin{pmatrix} a(x) & b_\nu(x) \\ \widetilde{b}_\nu(x) & \widetilde{a}(x) \end{pmatrix}, \quad B(x) = \begin{pmatrix} c(x) & d_\nu(x) \\ \widetilde{d}_\nu(x) & \widetilde{c}(x) \end{pmatrix}, \quad C(x) = \begin{pmatrix} 0 & d_\nu(x) \\ 5 & \widetilde{c}(x) \end{pmatrix}, \tag{3.50}$$

$$K(x,y) = \begin{pmatrix} k_0(x,y) & 0 \\ 0 & k_0(x,y) \end{pmatrix}, \quad k_0(x,y) = \frac{1}{\pi i}\frac{\left(\frac{|y|}{|x|}\right)^{\nu-1}\frac{\theta_0(y)}{\theta_0(x)} - 1}{y-x}. \tag{3.51}$$

The system (**??**) has to be considered in the space $L_p^\gamma \times L_p^\gamma$. The condrtion (2.3) for the kernel (**??**) is fulfilled because of the relation (3.5) if $-1 < \gamma < p-1$. The second condition (**??**), which is the condition of type (2.19) for the operator $S(S_\theta^{\nu-1} - S)$ is also satisfied by Corollary to Lemma **??**.

The conditions (**??**) and (2.4) being satisfied, we may calculate the Mellin transforms $\mathcal{K}_{++}^0(z)$ (see (2.7)) corresponding to our special kernel (3.51). To this end, we take into account the formula (2.24), and the relations

$$\left[\frac{\theta_0(y)}{\theta_1(x)}\right]_{++} = \left[\frac{\theta_0(y)}{\theta_0(x)}\right]_{--} = 1, \quad \left[\frac{\theta_0(y)}{\theta_0(x)}\right]_{+-} = -\frac{1}{\lambda^2}, \quad \left[\frac{\theta_0(y)}{\theta_0(x)}\right]_{-+} = -\lambda^4,$$

where the signs $\pm\pm$ mean that the point (x, y) belongs to the corresponding quadrant $R^2_{\pm\pm}$, and after easy calculations we obtain

$$\mathcal{K}_{++}^0(z) = -\frac{i\ sin\ \nu\pi}{sin\ z\pi\ sin\ (z+\nu)\pi}\ , \quad \mathcal{K}_{--}^0(z) = -\mathcal{K}_{++}^0(z), \tag{3.52}$$

$$\mathcal{K}_{+-}^0(z) = i\frac{\frac{1}{\lambda^2}sin\ z\pi - sin\ (z+\nu)\pi}{\sin z\pi\ sin\ (z+\nu)\pi}\ , \quad \mathcal{K}_{-+}^0(\xi) = -i\frac{\lambda^2 sin\ z\pi - sin\ (z+\nu)\pi}{\sin z\pi\ sin\ (z+\nu)\pi}\ . \tag{3.53}$$

We have to calculate the matrix symbols $\sigma_0(z)$ and $\sigma_\infty(z)$ defined in (2.13). The matrtx $\sigma_0(z)$ is reduced, after some simple transformations, to

$$\sigma_0(z) = \begin{pmatrix} I - \alpha_\lambda M_2(0) - \beta i M_1(0) & u_{6/\lambda} M_2(1) - i v_{1/\lambda} M_1(0) \\ u_\lambda M_2(0) + i v_\lambda M_2(0) & I - \alpha_{1/\lambda} M_2(0) + \beta i M_1(0) \end{pmatrix}, \tag{3.54}$$

where M_1 and M_2 are the matrices (2.29), while

$$\alpha_\lambda = \alpha_\lambda(z) = \frac{cos\ \nu\pi - \lambda^2}{sin\ z\pi\ sin\ (z+\nu)\pi}\ , \quad \beta = \beta(z) = \frac{sin\ \nu\pi}{sin\ z\pi\ sin\ (z+\nu)\pi}\ , \tag{3.55}$$

and

$$u_\lambda = u_\lambda(z) = \frac{cos\ (z+\nu)\pi - \frac{1}{\lambda^2} cos\ z\pi}{sin\ z\pi\ sin\ (z+\nu)\pi}\ , \quad v_\lambda = v_\lambda(z) = \frac{sin\ (z+\nu)\pi - \frac{1}{\lambda^2} sin\ z\pi}{sin\ z\pi\ sin\ (z+\nu)\pi}\ . \tag{3.56}$$

We need some properties of the functions (**??**) and (**??**), presented in the lemma below.

Lemma 3.10. *The following equalities hopd:*

$$v_\lambda u_{1/\lambda} - u_\lambda v_{1/\lambda} = \beta(\alpha_{1/\lambda} - \alpha_\lambda), \quad \alpha_\lambda \alpha_{1/\lambda} - u_\lambda u_{1/\lambda} = \alpha_\lambda + \alpha_{1/\lambda}, \tag{3.57}$$

$$\alpha_\lambda \alpha_{1/\lambda} - \beta^2 = \frac{cos\ \nu\pi}{sin\ z\pi\ sin\ (z+\nu)\pi} \left(\alpha_\lambda + \alpha_{1/\lambda}\right), \tag{3.58}$$

$$u_\lambda u_{1/\lambda} - v_\lambda v_{1/\lambda} = (\alpha_\lambda \alpha_{1/\lambda} - \beta^2)\ cos\ 2\pi z - \beta\left(\alpha_\lambda + \alpha_{1/\lambda}\right)\ sin\ 2\pi z. \tag{3.59}$$

Proof. The proof may be easily obtained directly in view of the connections

$$u_\lambda + i v_\lambda = (\alpha_\lambda + i\beta) e^{iz\pi}, \quad u_\lambda - i v_\lambda = (\alpha_\lambda - i\beta) e^{-iz\pi}.$$

□

b) Calculation of $\det \sigma_0(z)$ **and** $\det \sigma_\infty(z)$**.** We introduce the functions

$$\Delta_\pm(x) = \{a(x) \pm c(x)\}\{a[\tau(x)] \pm c[\tau(x)]\} - \{b(x) \pm d(x)\}\{b[\tau(x)] \pm d[\tau(x)]\}, \tag{3.60}$$

$$\Delta(x) = \{a(x) - c(x)\}\{a[\tau(x)] + c[\tau(x)]\} - \{b(x) + d(x)\}\{b[\tau(x)] - d[\tau(x)]\}. \tag{3.61}$$

Evidently,

$$\Delta_\pm(x) = \det\left[A(x) \pm B(x)\right]\ , \tag{3.62}$$

where $A(x)$ and $B(x)$ are the matrices defined in (3.50).

Everywhere below $\Delta_\pm(\delta)$ stands for $\lim_{\delta\to 0} \Delta_\pm(x)$ and similarly for $\Delta(\delta)$.

Lemma 3.11. *Let* $\Delta_\pm(\delta) \neq 0$*. Then the determinant of the matrix (**??**) is calculated by the formula*

$$\det \sigma_0(z) = 1 - \frac{cos\ \nu\pi - \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)}{2sin\ z\pi\ sin\ (z+\nu)\pi} + \frac{cos\ \nu\pi - \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right)}{2sin\ z\pi\ sin\ (z+\nu)\pi} \frac{\Delta(\delta)\Delta(\infty)}{\Delta_+(\delta)\Delta_-(\delta)}, \tag{3.63}$$

and also

$$\det \sigma_\infty(z) = \det \sigma_0(z). \tag{3.64}$$

Proof. To calculate the determinant of the matrix (**??**), we first transform the block $\alpha_\lambda M_2 + \beta i M_1$ as follows: $\alpha_\lambda M_2 + \beta i M_1 = \left[\frac{\alpha_\lambda+\beta i}{2}(A+B)^{-1} - \frac{\alpha_\lambda-\beta i}{2}(A-B)^{-1}\right] C$. We represent the matrix C as $C = \left[\frac{1}{2}(A+B) - \frac{1}{2}(A-B)\right] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, which yields

$$\alpha_\lambda M_2 + \beta i M_1 = \frac{1}{2}\left[\alpha_\lambda I - \frac{\alpha_\lambda + \beta i}{2}(A+B)^{-1}(A-B) - \frac{\alpha_\lambda - \beta i}{2}(A-B)^{-1}(A+B)\right] \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

After easy calculations we arrive at

$$\alpha_\lambda M_2 + \beta i M_1 = \begin{pmatrix} 0 & -\left(\frac{\alpha_\lambda+\beta i}{2\Delta_+} - \frac{\alpha_\lambda-\beta i}{2\Delta_-}\right)(b_\nu \widetilde{c} - d_\nu \widetilde{a}) \\ \\ 0 & \frac{\alpha_\lambda}{2} - \frac{\alpha_\lambda+\beta i}{4\Delta_+}\widetilde{\Delta} - \frac{\alpha_\lambda-\beta i}{4\Delta_-}\Delta \end{pmatrix},$$

where $\Delta_\pm, \Delta$ are the functions (3.60)-(3.61).

In a similar way the blocks $\alpha_{1/\lambda} M_2 - \beta i M_1$ and $u_\lambda M_2 + i v_\lambda M_1$ and $u_{1/\lambda} M_2 - i v_{1/\lambda} M_1$ are transformed and as a result, the blocs $\sigma^{kj}(z)$, defined in (2.13), in the case of our matrix (**??**) take the form:

$$\sigma^{11}(z) = \begin{pmatrix} 1 & (b_\nu \widetilde{c} - d_\nu \widetilde{a})\left(\frac{\alpha_\lambda+i\beta}{2\Delta_+} - \frac{\alpha_\lambda-i\beta}{2\Delta_-}\right) \\ \\ 0 & 1 - \frac{\alpha_\lambda}{2} + \frac{\alpha_\lambda+i\beta}{4\Delta_+}\widetilde{\Delta} + \frac{\alpha_\lambda-i\beta}{4\Delta_-}\Delta \end{pmatrix},$$

$$\sigma^{12}(z) = \begin{pmatrix} 0 & -(b_\nu \widetilde{c} - d_\nu \widetilde{a})\left(\frac{u_{1/\lambda}-iv_{1/\lambda}}{2\Delta_+} - \frac{u_{1/\lambda}+iv_{1/\lambda}}{2\Delta_-}\right) \\ \\ 0 & \frac{u_{1/\lambda}}{2} - \frac{u_{1/\lambda}-iv_{1/\lambda}}{4\Delta_+}\widetilde{\Delta} - \frac{u_{1/\lambda}+iv_{1/\lambda}}{4\Delta_-}\Delta \end{pmatrix},$$

$$\sigma^{21}(z) = \begin{pmatrix} 0 & -(b_\nu \widetilde{c} - d_\nu \widetilde{a})\left(\frac{u_\lambda+iv_\lambda}{2\Delta_+} - \frac{u_\lambda-iv_\lambda}{2\Delta_-}\right) \\ \\ 0 & \frac{u_\lambda}{2} - \frac{u_\lambda+iv_\lambda}{4\Delta_+}\widetilde{\Delta} - \frac{u_\lambda-iv_\lambda}{4\Delta_-}\Delta \end{pmatrix},$$

$$\sigma^{22}(z) = \begin{pmatrix} 1 & (b_\nu \widetilde{c} - d_\nu \widetilde{a})\left(\frac{\alpha_{1/\lambda}-i\beta}{2\Delta_+} - \frac{\alpha_{1/\lambda}+i\beta}{2\Delta_-}\right) \\ \\ 0 & 1 - \frac{\alpha_{1/\lambda}}{2} + \frac{\alpha_{1/\lambda}-i\beta}{4\Delta_+}\widetilde{\Delta} + \frac{\alpha_{1/\lambda}+i\beta}{4\Delta_-}\Delta \end{pmatrix}.$$

In these formulas $\alpha_\lambda, \beta_\lambda, u_\lambda$ and v_λ depend on z, while a, b_ν, c and d_ν are functions of x and we obtain the blocks $\sigma_0^{kj}(z)$, calculating $\sigma^{kj}(z)$ at the point $x = \delta$, and the blocks

$\sigma_\infty^{kj}(\xi)$ calculating them at the point $x=\infty$. We have

$$\det\begin{pmatrix}\sigma^{11}(z) & \sigma^{12}(z)\\ \\ \sigma^{21}(z) & \sigma^{22}(z)\end{pmatrix}=\begin{vmatrix}1-\frac{\alpha_\lambda}{2}+\frac{\alpha_\lambda+i\beta}{4\Delta_+}\widetilde{\Delta}+\frac{\alpha_\lambda-i\beta}{4\Delta_-}\Delta & \frac{u_{1/\lambda}}{2}-\frac{u_{1/\lambda}-iv_{1/\lambda}}{4\Delta_+}\widetilde{\Delta}-\frac{u_{1/\lambda}+iv_{1/\lambda}}{4\Delta_-}\Delta\\ \\ \frac{u_\lambda}{2}-\frac{u_\lambda+iv_\lambda}{4\Delta_+}\widetilde{\Delta}-\frac{u_\lambda-iv_\lambda}{4\Delta_-}\Delta & 1-\frac{\alpha_{1/\lambda}}{2}+\frac{\alpha_{1/\lambda}-i\beta}{4\Delta_+}\widetilde{\Delta}+\frac{\alpha_{1/\lambda}+i\beta}{4\Delta_-}\Delta\end{vmatrix}.$$

By direct calculations we obtain

$$det\ \sigma_0(z)=1-\tfrac{1}{2}\left(\alpha_\lambda+\alpha_{1/\lambda}\right)+\tfrac{1}{4}\left(\alpha_\lambda\alpha_{1/\lambda}-u_\lambda u_{1/\lambda}\right)$$

$$+\tfrac{\widetilde{\Delta}}{\Delta_+}\left\{\tfrac{1}{4}\left[\alpha_\lambda+\alpha_{1/\lambda}-\alpha_\lambda\alpha_{1/\lambda}+u_\lambda u_{1/\lambda}\right]+\tfrac{i}{8}\left[v_\lambda u_{1/\lambda}-u_\lambda v_{1/\lambda}-\beta(\alpha_{1/\lambda}-\alpha_\lambda)\right]\right\}$$

$$+\tfrac{\Delta}{\Delta_-}\left\{\tfrac{1}{4}\left[\alpha_\lambda+\alpha_{1/\lambda}-\alpha_\lambda\alpha_{1/\lambda}+u_\lambda u_{1/\lambda}\right]-\tfrac{i}{8}\left[v_\lambda u_{1/\lambda}-u_\lambda v_{1/\lambda}-\beta(\alpha_{1/\lambda}-\alpha_\lambda)\right]\right\}$$

$$+\tfrac{1}{16}\left(\tfrac{\widetilde{\Delta}}{\Delta_+}\right)^2\left[(\alpha_\lambda+i\beta)(\alpha_{1/\lambda}-i\beta)-(u_\lambda+iv_\lambda)(u_{1/\lambda}-iv_{1/\lambda})\right]$$

$$+\tfrac{1}{16}\left(\tfrac{\Delta}{\Delta_-}\right)^2\left[(\alpha_\lambda-i\beta)(\alpha_{1/\lambda}+i\beta)-(u_\lambda-iv_\lambda)(u_{1/\lambda}+iv_{1/\lambda})\right]$$

$$+\tfrac{1}{8}\tfrac{\widetilde{\Delta}\Delta}{\Delta_+\Delta_-}\left[\alpha_\lambda\alpha_{1/\lambda}-\beta^2-u_\lambda u_{1/\lambda}+v_\lambda v_{1/\lambda}\right].$$

By Lemma **??** this reduces to

$$\det\sigma_0(z)=1-\frac{\alpha_\lambda+\alpha_{1/\lambda}}{4}+\frac{\alpha_\lambda+\alpha_{1/\lambda}}{4}\frac{\widetilde{\Delta}\Delta}{\Delta_+\Delta_-} \tag{3.65}$$

which coincides with (3.63).

Finally, the relation (3.64) becomes obvious, if we note that $\Delta(\delta)=\widetilde{\Delta}(\infty)$ and $\Delta_\pm(\delta)=\Delta_\pm(\infty)$. □

We have to calculate $\det\sigma_0(z)$ along the vertical line $z=i\xi-1+\frac{1+\gamma}{p},\quad \xi\in R^1$.

Lemma 3.12. *Let $\Delta_\pm(\delta)\neq 0$. Then*

$$\det\sigma_0\left(i\xi-1+\frac{1+\gamma}{p}\right)=1+\frac{cos\ \nu\pi-\frac{1}{2}\left(\lambda^2+\frac{1}{\lambda^2}\right)}{2\left[\sin^2\frac{\nu\pi}{2}+sh^2\ \xi\pi\right]}\left(1-\frac{\Delta(\delta)\Delta(\infty)}{\Delta_+(\delta)\Delta_-(\delta)}\right). \tag{3.66}$$

Proof. The proof is an immediate consequence of Lemma 3.11 if we take into account the relation $sin\ z\pi sin\ (z+\nu)\pi\Big|_{z=i\xi+1-\frac{1+\gamma}{p}}=-\sin^2\frac{\nu\pi}{2}-sh^2\ \xi\pi.$ □

c) The result on Fredholmness. To formulate the main result, we introduce the notation for the following ray in the complex plane

$$\mathfrak{L}_{\lambda;p,\gamma}=\left\{z:\ z=\frac{1-\mu}{cos\ \nu\pi-\mu}+\frac{t}{cos\ \nu\pi-\mu},\quad t\in[0,+\infty)\right\}, \tag{3.67}$$

where

$$\mu = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right), \ \lambda \in \mathbf{C}\backslash\{0\}, \quad \nu = \frac{2(1+\gamma)}{p} \tag{3.68}$$

and it is assumed that $\lambda \neq e^{\pm\frac{\nu\pi i}{2}}$, which is equivalent to $cos\ \nu\pi - \mu \neq 0$. In the case $\lambda = e^{\pm\frac{\nu\pi i}{2}}$ we put

$$\mathfrak{L}_{\lambda;p,\gamma} = \emptyset. \tag{3.69}$$

Theorem 3.13. *Let the shift $\tau(x)$ preserve the orientation on R^1 and let $a(x),\ c(x) \in C(\dot{R}^1)$ and for some $\lambda \in \mathbf{C}\backslash\{0\}$*

$$b(x)|x-\delta|^\nu \left[\lambda\theta_+(x-\delta) + \frac{1}{\lambda}\theta_-(x-\delta)\right],\ d(x)|x-\delta|^\nu \left[\lambda\theta_+(x-\delta) + \frac{1}{\lambda}\theta_-(x-\delta)\right] \in C(\dot{R}^1). \tag{3.70}$$

I. In the case $\lambda \neq e^{\pm i\frac{\nu\pi}{2}}$, the operator (3.32) is Fredholm in the space $L_p^\gamma(R^1), 1 < p < \infty, -1 < \gamma < p-1,$ if and only if

$$1) \qquad \inf_{x\in\dot{R}^1} |\Delta_\pm(x)| \neq 0,\ x \in \dot{R}^1 \tag{3.71}$$

and

$$2) \qquad \frac{\Delta(\delta)\Delta(\infty)}{\Delta_+(\delta)\Delta_-(\delta)} \notin \mathfrak{L}_{\lambda;p,\gamma}, \tag{3.72}$$

the condition (3.72) being equivalent to

$$\frac{\Delta(\delta)\Delta(\infty)}{\Delta_+(\delta)\Delta_-(\delta)}(\cos\nu\pi - \mu) + \mu - 1 \notin [0,\infty), \quad \mu = \frac{1}{2}\left(\lambda^2 + \frac{1}{\lambda^2}\right). \tag{3.73}$$

II. In the case $\lambda = e^{\pm i\frac{\nu\pi}{2}}$ the Fredholmness of the operator (3.32) is equivalent to the condition (3.71) only, the ray $\mathfrak{L}_{\lambda;p,\gamma}$ being empty in this case.

The formula for the index in all the cases is

$$Ind_{L_p^\gamma}\ K = \frac{1}{2} ind\ \frac{\Delta_-(x)}{\Delta_+(x)}\ . \tag{3.74}$$

Proof. According to Subsection 2.2, Fredholmness of the operator (3.32) is equivalent to that of the matrix operator (3.46) with formula (**??**) for the index. The matrix operator (3.46) has the form (**??**) and its Fredholmness is covered by Theorem 2.7, so that the condition $\Delta_\pm(x) \neq 0$ must be fulfilled for the operator (3.32) to be Fredholm. Then Lemma 3.11 is applicable, from which one can easily derive that the condition $\det\sigma_0\left(i\xi + 1 - \frac{1+\gamma}{p}\right) \neq 0$ of Theorem 2.7 is equivalent to (3.72). It remains to note that the case $\lambda = e^{\pm i\frac{\nu\pi}{2}}$ is degenerate in a sense, see (3.69).

Formula (3.74) follows from (2.25) because of the relations (3.62) and (3.64). □

In the following corollary to Theorem 3.13 we single out the most interesting cases of the choice of the parameter λ.

Corollary 1. *Let the shift $\tau(x)$ preserve the orientation on R^1.*
a) Let $\lambda = 1$. Then the conditions (3.70) take the form $b(x)|x-\delta|^\nu,\ d(x)|x-\delta|^\nu \in C(\dot{R}^1)$ and the additional condition (3.72) is

$$\frac{\Delta(\delta)\Delta(\infty)}{\Delta_+(\delta)\Delta_-(\delta)} \notin (-\infty, 0] \ . \tag{3.75}$$

b) Let $\lambda = i$. Then the conditions (3.70) take the form $b(x)|x-\delta|^\nu sign\ (x-\delta),\ d(x)|x-\delta|^\nu sign\ (x-\delta) \in C(\dot{R}^1)$ and the additional condition (3.72) is

$$\frac{\Delta(\delta)\Delta(\infty)}{\Delta_+(\delta)\Delta_-(\delta)} \notin \left[\frac{1}{cos^2\ \frac{\pi}{p}(\gamma+1)}, +\infty\right) \ . \tag{3.76}$$

c) Let $\lambda = e^{\frac{\pm i\nu\pi}{2}}$. Then correspondingly $b(x)[\mp(x-\delta)]^\nu,\ d(x)[\mp(x-\delta)]^\nu \in C(\dot{R}^1)$ with $(\pm x)^\nu = |x|^\nu[\theta_\pm(x) + e^{i\nu\pi}\theta_\mp(x)]$, and the condition (3.72) is fulfilled automatically.

Remark 3.14. *The point $z = 1$ never belongs to the ray $\mathfrak{L}_{\lambda;p,\gamma}$, whatever λ, p and γ are.*

What is the sense of a possibility to choose different values of λ?

Let $b_0(x) = b(x)|x-\delta|^\nu$ and $d_0(x) = d(x)|x-\delta|^\nu$. Our assumptions on these coefficients: $b_0(x)\theta(x), d_0(x)\theta(x) \in C(\dot{R}^1)$, where $\theta(x) = \lambda\theta_+(x-\delta)+\frac{1}{\lambda}\theta_-(x-\delta)$, mean that the functions $b_0(x)$ and $d_0(x)$ themselves are piece-wise continuous with jumps only at the points $x = \delta$ and $x = \infty$. Easy calculation gives $\lambda b_0(\delta+0) = \frac{1}{\lambda}b_0(\delta-0)$ and $\lambda b_0(+\infty) = \frac{1}{\lambda}b_0(-\infty)$ and similarly for $d_0(x)$. We arrive at the requirement that the jumps of the functions $b_0(x)$ and $d_0(x)$ at the points $x = 0$ and $x = \infty$ must be coordinated:

$$\frac{b_0(\delta-0)}{b_0(\delta+0)} = \frac{b_0(\infty)}{b_0(-\infty)} = \frac{d_0(\delta-0)}{d_0(\delta+0)} = \frac{d_0(\infty)}{d_0(-\infty)}$$

and then $\lambda = \sqrt{\frac{b_0(\delta-0)}{b_0(\delta+0)}}$, where we suppose for simplicity that the corresponding numbers are different from zero.

Hence, the choice $\lambda = 1$ e.g. means that the functions $b_0(x)$ and $d_0(x)$ are continuous at the points $x = 0$ and $x = \infty$. In the general case we may read the condition (3.73) directly in terms of the coefficients, avoiding usage of the parameter λ:

$$\mu = \frac{1}{2}\left[\frac{b_0(\delta-0)}{b_0(\delta+0)} + \frac{b_0(\delta+0)}{b_0(\delta-0)}\right].$$

3.6 The case of change of the orientation ($D > 0$).

The arguments are similar, so that we dwell only on some slight differences in calculations.

The involutive operator (3.2) may be introduced now only via two possibilities given in (3.3). When $\theta(x) = sign\ x$ or $\theta(x) = 1$, the corresponding matrix operator (3.48) has the form

$$\mathbb{K} = \begin{pmatrix} aI + cS & b_\nu I - d_\nu S^{\nu-1} \\ \\ \widetilde{b}_\nu I + \widetilde{d}_\nu S & \widetilde{a}I - \widetilde{c}S^{\nu-1} \end{pmatrix}, \qquad \mathbb{K} = \begin{pmatrix} aI + cS & b_\nu I - d_\nu S_{\nu-1} \\ \\ \widetilde{b}_\nu I + \widetilde{d}_\nu S & \widetilde{a}I - \widetilde{c}S_{\nu-1} \end{pmatrix}, \tag{3.77}$$

respectively. The first equality in (3.77) is obtained by means of the relation (3.9), and the second one - by means of (3.27) under the choice $\alpha = 0$.

Therefore, the operator $\mathbb{K}$ now has the form (**??**) with

$$A(x) = \begin{pmatrix} a(x) & b_\nu(x) \\ \\ \widetilde{b}_\nu(x) & \widetilde{a}(x) \end{pmatrix}, \quad B(x) = \begin{pmatrix} c(x) & -d_\nu(x) \\ \\ \widetilde{d}_\nu(x) & -\widetilde{c}(x) \end{pmatrix}, \quad C(x) = \begin{pmatrix} 0 & -d_\nu(x) \\ \\ 0 & -\widetilde{c}(x) \end{pmatrix}$$

and $K(x, y)$ as in (3.51) where

$$k_0(x,y) = \frac{1}{\pi i} \frac{\left(\frac{|y|}{|x|}\right)^{\nu-1} - 1}{y - x} \quad \text{and} \quad k_0(x,y) = \frac{1}{\pi i} \frac{\left(\frac{|y|}{|x|}\right)^{\nu-1} \frac{sign\ y}{sign\ x} - 1}{y - x}$$

correspondingly to the cases $\theta(x) = sign\ x$ and $\theta(x) = 1$.

Therefore, it is easily seen that now we have a symbol similar to that we had in the case $D < 0$, with the only difference that now the quotients $\frac{\widetilde{\Delta}}{\Delta_+}$ and $\frac{\Delta}{\Delta_-}$ should be replaced by $\frac{\Delta_+}{\widetilde{\Delta}}$ and $\frac{\Delta_-}{\Delta}$, respectively. As a result, we arrive at the following theorem.

Theorem 3.15. *Let the shift $\tau(x)$ change the orientation on R^1 and let one of the following assumptions be satisfied:*

$$a(x), b(x)|x - \delta|^\nu sign\ (x - \delta), c(x), d(x)|x - \delta|^\nu sign\ (x - \delta) \ \in \ C(\dot{R}^1) \tag{3.78}$$

or

$$a(x), b(x)|x - \delta|^\nu, c(x), d(x)|x - \delta|^\nu \ \in \ C(\dot{R}^1). \tag{3.79}$$

The operator of the form (3.32) is Fredholm in the space $L_p^\gamma(R^1), 1 < p < \infty, -1 < \gamma < p - 1$, if and only if

$$\inf_{x \in \dot{R}^1} |\Delta(x)| \neq 0, \quad x \in \dot{R}^1 \tag{3.80}$$

and

$$\frac{\Delta_+(\delta)\Delta_-(\delta)}{\Delta(\delta)\Delta(\infty)} \notin \left[\frac{1}{cos^2\ \frac{\pi}{p}(\gamma+1)}, +\infty\right) \tag{3.81}$$

in the case (3.78) and

$$\frac{\Delta_+(\delta)\Delta_-(\delta)}{\Delta(\delta)\Delta(\infty)} \notin (-\infty, 0] \tag{3.82}$$

in the case (3.79). Under these conditions

$$Ind_{L_p^\gamma}\ K = ind\ \Delta(x). \tag{3.83}$$

3.7 The case of a non fractional-linear shift on R^1.

Let $\tau(x)$ be a one-to-one mapping of R^1 onto itself, differentiable everywhere, except for the singular point $\delta = \tau(\infty)$, where it has a singularity. We suppose that the following conditions are fulfilled:

1) $\tau[\tau(x)] = x, \quad \tau(x) \not\equiv x$;
2) $\tau'(x)(x-\delta)^2 \in H^\lambda(\dot{R}^1)$ and $\lim_{x\to\infty} \tau'(x)x^2 \neq 0$, under the assumption that $\delta \neq \infty$.

We recall that a function $f(x) \in H^\lambda(\dot{R}^1)$, $0 < \lambda \leq 1$, if $f(x) \in C(\dot{R}^1)$ and $|f(x_1) - f(x_2)| \leq C|x_1 - x_2|^\lambda (1+|x_1|)^{-\lambda}(1+|x_2|)^{-\lambda}$, $\quad x_1, x_2 \in R^1$.

In the case $\delta = \infty$, the condition 2) should be replaced by the condition $2'$) $\tau'(x) \in H^\lambda(\dot{R}^1)$.

The operator A is to be considered in the same weighted space $L_p^\gamma(R^1)$ with the weight-function $|x-\delta|^\gamma$ fixed to the singular point $\delta = \tau(\infty)$.

The main arguments are the same, so we dwell only on some principal points. The involutive operator Q_ν given before by (3.2) is now introduced as $(Q_\nu\varphi)(x) = |\tau'(x)|^{\frac{\nu}{2}}\varphi(x)$. We omit the generalization involving the function $\theta(x)$ as in (3.2).

Lemma 3.16. *The operator Q_ν is involutive for any ν. It is bounded in the space $L_p^\gamma(R^1)$, $1 \leq p \leq \infty$, $-\infty < \gamma < \infty$, if $\nu = \frac{2}{p}(1+\gamma)$.*

Proof. The proof is direct and is based on the assumptions 1)-2). □

Lemma 3.17. *Let $1 < p < \infty$, $-1 < \gamma < p-1$ and $\nu = \frac{2}{p}(1+\gamma)$. The relations between the operator Q_ν and the operators S^α and S_α, defined in (3.8), are given by the same formulas (3.9) - (3.10), up to operators compact in the space $L_p^\gamma(R^1)$, if we replace sign $(-D)$ there by sign $[\tau'(x)]$ and take $\frac{1+\gamma}{p} - 1 < \alpha < \frac{1+\gamma}{p}$.*

We refer to [3] for the proof of Lemma 3.17.

Because of Lemma 3.17, the further investigation based on the application of Theorem **??** is more or less similar to what we did in the previous subsections. By this reason, we only explain how we construct the operator U from Axiom 1 of Subsection 3.3 and formulate the main result in Theorem 3.18 below, referring for details to the paper [3].

The operator U in this case is defined as follows:

$$U\varphi \;=\; u_1(x)\varphi(x), \quad \text{if} \quad \tau'(x) > 0, \tag{3.84}$$

$$U\varphi \;=\; u_2(x)\varphi(x) + i\; v(x)\left(S^{\frac{\nu-1}{2}}\varphi\right)(x), \quad \text{if} \quad \tau'(x) < 0, \tag{3.85}$$

where $u_j(x) = \frac{x-\tau(x)}{x+\tau(x)-2a_j}$, $\quad j = 1, 2$, with $a_1 = i$ and $a_2 = \tau(\infty)$ and $\;(x) = e^{\tau'(x)+\tau'[\tau(x)]}$ and for simplicity we assume that $\delta \neq \infty$. We remark that $u_1(x), u_2(x), v(x) \in C(\dot{R}^1)$ and $u_j[\tau(x)] = -u_j(x)$, $\quad j = 1, 2$, and $\quad u_1(x) \neq 0, x \in \dot{R}^1$, since a shift preserving the orientation has no fixed points, while real-valued coefficients $u_2(x)$ and $v(x)$ vanish at different points. It is also easy to see that in the case when the shift changes the orientation, $|u_2(x)| \leq 1$ and $u_2(\delta) = -1$ and $u_2(\infty) = 1$.

Theorem 3.18. *Let $a(x), |\tau'(x)|^{-\frac{1+\gamma}{p}}b(x), c(x), |\tau'(x)|^{-\frac{1+\gamma}{p}}d(x) \in C(\dot{R}^1)$ and let $\delta = \tau(\infty)$. The operator (3.32) with a Carleman shift satisfying the assumptions 1)-2), is Fredholm in the space $L_p^\gamma(\dot{R}^1), 1 < p < \infty, -1 < \gamma < p-1$, if and only if the assumptions (3.71) and (3.76) in the case of $\tau'(x) > 0$ and (3.80)and (3.81) in the case of $\tau'(x) < 0$ are satisfied. It has the same formulas (3.74), (3.83) for the index.*

4 On singular integral equations with Carleman shift in weighted spaces

a) Application of general theorem 3.9. Let Γ be a closed bounded Lyapunov curve and $S = S_\Gamma$ the singular operator along Γ. We consider an arbitrary Carleman shift $\tau(t)$ on Γ such that $\tau'(t) \in H^\lambda(\Gamma)$ for some $\lambda \in (0, 1]$ and treat the operator

$$K\varphi = a(t)\varphi(t) + b(t)\varphi[\tau(t)] + c(t)(S\varphi)(t) + d(t)(S\varphi)[\tau(t)], \quad t \in \Gamma \tag{4.1}$$

in the weighted space $L_p(\Gamma, \rho) = \{f : \ \|f\|_{L_p(\Gamma,\rho)} < \infty\}$ with

$$\|f\|^p_{L_p(\Gamma,\rho)} = \int_\Gamma |f(t)|^p \rho(t)|dt| < \infty, \quad \rho(t) \geq 0.$$

The shift operator being not bounded in $L_p(\Gamma, \rho)$ in general, we use the familiar idea of its modification in the form

$$(Q\varphi)(t) = m(t)\varphi[\tau(t)], \tag{4.2}$$

where $m(t)$ is to be chosen in such a way that $Q^2 = I$ and Q is bounded in $L_p(\Gamma, \rho)$, compare with (3.2) or (3.7).

Lemma 4.1. *Let $1 \leq p \leq \infty$ and non-negative measurable function $\rho(t)$ vanish on a set of Γ of measure 0. The operator Q is bounded in the space $L_p(\Gamma, \rho)$ if and only if*

$$m(t) = c(t) \left[\frac{\rho[\tau(t)]}{\rho(t)}\right]^{\frac{1}{p}}, \tag{4.3}$$

where $c(t)$ is any non-negative measurable bounded function. Under the choice (4.3) the operator Q is involutive if and only if $c(t)c[\tau(t)] \equiv 1$. Under the choice $c(t) = |\tau'(t)|^{\frac{1}{p}}$ we also have $\|Q\varphi\| = \|\varphi\|$.

Proof is direct.

In the proof of Theorem 4.2 below for simplicity we take $c(t) \equiv 1$ in (4.3) so that $(Q\varphi)(t) = \left[\frac{\rho[\tau(t)]}{\rho(t)}\right]^{\frac{1}{p}} \varphi[\tau(t)]$. We denote

$$b^*(t) = \left[\frac{\rho(t)}{\rho[\tau(t)]}\right]^{\frac{1}{p}} b(t), \quad d^*(t) = \left[\frac{\rho(t)}{\rho[\tau(t)]}\right]^{\frac{1}{p}} d(t)$$

and suppose that

$$a(t), \ b^*(t), \ c(t), \ d^*(t) \ \in C(\Gamma). \tag{4.4}$$

In theorem 4.2, $A_p(\Gamma)$ stands for the Muckenhoupt class of weight functions, see e.g. [1], p. 28. By $\widetilde{a}(t)$ we denote $\widetilde{a}(t) = a[\tau(t)], etc$

Theorem 4.2. *Let $\rho(t) \in A_p(\Gamma)$, $1 < p < \infty$, and $\tau(t)$ a Carleman shift preserving the orientation on Γ. Under assumptions (4.4) the operator K is Fredholm in the space*

$L_p(\Gamma,\rho)$ if and only if the matrix operator

$$\mathbb{K} = \begin{pmatrix} aI + cS & \widetilde{b}^* I + \widetilde{d}^* S_\rho \\ b^* I + d^* S & \widetilde{a} I + \widetilde{c} S_\rho \end{pmatrix}, \tag{4.5}$$

in Fredholm in $L_p^2(\Gamma,\rho) = L_p(\Gamma,\rho)\times L_p(\Gamma,\rho)$, where

$$S_\rho\varphi = \frac{1}{\pi i}\int_\Gamma \left(\frac{\rho(t)\rho[\tau(w)]}{\rho(w)\rho[\tau(t)]}\right)^{\frac{1}{p}} \frac{\varphi(w)\,dw}{w-t}$$

and then $Ind\ K = \frac{1}{2} Ind\ \mathbb{K}$.

Proof. We represent the operator K in the form $K = A_1 + QA_2$, where $A_1 = aI + cS$ and $A_2 = b^*I + d^*S$ and apply Theorem **??**. The operator U, required by Axiom 1 may be taken as $U\varphi = [t-\alpha(t)]\varphi(t)$ and then Axiom 1 and 2 are satisfied.

The application of the Theorem **??** leads to the matrix operator $\mathbb{K} = \begin{pmatrix} A_1 & QA_2Q \\ A_2 & QA_1Q \end{pmatrix}$. Calculating its entires we arrive at the operator (4.5). □

We note that the matrix operator obtained in (4.5) is a singular-type matrix operator without shift, but with unbounded coefficients.

b) An analogue of the equation (A) on the unit circle. The operator (A) may be easily transformed to some operator with shift on the unit circle $\Gamma = \{t : \ |t| = 1\}$ by the standard change of variables $\frac{x-i}{x+i} = t, \quad x = i\frac{1+t}{1-t}$. The shift $\tau(x) = \frac{\delta x + \beta}{x-\delta}$ is then transformed into the Carleman shift

$$\alpha(t) = \frac{t-\mu}{\overline{\mu}t - 1}, \quad |t| = 1; \qquad \mu = \frac{\beta + 1 + 2i\delta}{\beta - 1}, \quad \beta \neq 1$$

on the unit circle: $\varphi[\tau(x)] = \Phi[\alpha(t)]$, where $\Phi(t) = \varphi\left(i\frac{1+t}{1-t}\right)$. In the case $\beta = 1$ one has $\alpha(t) = \frac{t_0}{t}$ where $t_0 = \frac{\delta - i}{\delta + i} = \alpha(1)$. It is easily checked that

$$\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(s)ds}{s-x} = \frac{1}{\pi i}\int_\Gamma \frac{1-t}{1-w}\frac{\Phi(w)dw}{w-t} \quad \text{and} \quad \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\varphi(s)ds}{s-\tau(x)} = \frac{1}{\pi i}\int_\Gamma \frac{1-\alpha(t)}{1-w}\frac{\Phi(w)dw}{w-\alpha(t)},$$

so that the equation $K\varphi = f$ generated by the equation (A) is transformed into the equation

$$a_0(t)\psi(t) + b_0(t)\psi[\alpha(t)] + \frac{c_0(t)}{\pi i}\int_\Gamma \frac{\psi(w)dw}{w-t} + \frac{d_0(t)}{\pi i}\int_\Gamma \frac{\psi(w)dw}{w-\alpha(t)} = g(t) \tag{4.6}$$

where $\psi(t) = \frac{1}{1-t}\varphi\left(i\frac{1+t}{1-t}\right)$ and similarly for $g(t)$ in terms of $f(x)$, and $a_0(t) = a\left(i\frac{1+t}{1-t}\right)$, $b_0(t) = \frac{1-\alpha(t)}{1-t} b\left(i\frac{1+t}{1-t}\right)$, $c_0(t) = c\left(i\frac{1+t}{1-t}\right)$, $d_0(t) = \frac{1-\alpha(t)}{1-t} d\left(i\frac{1+t}{1-t}\right)$. Evidently,

$$\|\varphi\|^p_{L_p^\gamma(R^1)} = 2\sqrt{\delta^2+1}\int_\Gamma |\psi(t)|^p |t-t_0|^\gamma |t-1|^{p-2-\gamma}|dt|.$$

Therefore, if the solutions of the equation $K\varphi = f$ were considered in the space $L_p^\gamma(R^1)$, solutions $\psi(t)$ of the equation (4.6) must be looked for in the weighted space

$$L_p(\Gamma,\rho) = \left\{\psi : \int_\Gamma |\psi(t)|^p \rho(t)|dt| < \infty\right\}, \quad \rho(t) = |t-t_0|^\gamma |t-1|^{p-2-\gamma}.$$

This weight is not invariant with respect to the shift $\alpha(t)$ (except for the case $\gamma = \frac{p}{2} - 1$), so we have to put some assumptions on the coefficients $b_0(t)$ and $d_0(t)$, according to (4.4). It is convenient to put $c(t) = \frac{\alpha(t)-1}{|\alpha(t)-1|}\frac{|t-1|}{t-1}$ in (4.3), so that $|c(t)| \equiv 1$ and $c(t)c[\alpha(t)] \equiv 1$. After easy calculations we see that the requirements (4.4) for $b\left(i\frac{1+t}{1-t}\right)$ and $d\left(i\frac{1+t}{1-t}\right)$ take the form $\left|\frac{t-t_0}{t-1}\right|^\nu b\left(i\frac{1+t}{1-t}\right)$, $\left|\frac{t-t_0}{t-1}\right|^\nu d\left(i\frac{1+t}{1-t}\right) \in C(\Gamma)$ which corresponds to the requirements $|x-\delta|^\nu b(x),\ |x-\delta|^\nu d(x) \in C(\dot{R}^1)$, the latter meaning the choice $\lambda = 1$ in (3.40).

Using the results for the equation (A), namely, Theorem 3.13 (and its Corollary 1) and Theorem 3.15, we may formulate the corresponding statements for the equation (4.6). It is clear that instead of the points $t = t_0$ and $t = 1$ one may take arbitrary points t_1 and $t_2 = \alpha(t_1)$ and consider the weight

$$\rho(t) = |t-t_1|^\gamma |t-t_2|^{p-2-\gamma}, \quad t_2 = \alpha(t_1), \quad t_1 \in \Gamma, \quad -1 < \gamma < p-1. \tag{4.7}$$

In the space $L_p(\Gamma,\rho)$ we consider the equation (4.1) on the unit circle Γ, where $\alpha(t) = \frac{t-\mu}{\bar{\mu}t-1}$ is a fractional-linear Carleman shift on Γ with $|\mu| \neq 1$. We denote, as in (3.60)-(3.61):

$$\Delta_\pm(t) = \{a(t) \pm c(t)\}\{a[\alpha(t)] \pm c[\alpha(t)]\} - \{b(t) \pm d(t)\}\{b[\alpha(t)] \pm d[\alpha(t)]\}, \quad t \in \Gamma, \tag{4.8}$$

$$\Delta(t) = \{a(t) - c(t)\}\{a[\alpha(t)] + c[\alpha(t)]\} - \{b(t) + d(t)\}\{b[\alpha(t)] - d[\alpha(t)]\}, \quad t \in \Gamma. \tag{4.9}$$

Applying Theorem 3.13 (and its Corollary 1) and Theorem 3.15, we arrive at the following theorem.

Theorem 4.3. *Let*

$$a(t), \quad \left|\frac{t-t_1}{t-\alpha(t_1)}\right|^\nu b(t), \quad c(t), \quad \left|\frac{t-t_1}{t-\alpha(t_1)}\right|^\nu d(t) \ \in \ C(\Gamma). \tag{4.10}$$

The operator K of the form (4.1) is Fredholm in the space $L_p(\Gamma,\rho)$ with the weight (4.7) if and only if

$$\frac{\Delta(t_1)\Delta(t_2)}{\Delta_+(t)\Delta_-(t)} \notin (-\infty, 0] \tag{4.11}$$

and

$$\inf_{t\in\Gamma} |\Delta_\pm(t)| \neq 0, \quad \textit{in the case } |\mu| < 1 \ \ (\textit{preservation of the orientation})$$

$$\inf_{t\in\Gamma} |\Delta(t)| \neq 0, \quad \textit{in the case } |\mu| > 1 \ \ (\textit{change of the orientation}).$$

Then Ind $K = \frac{1}{2}$ind $\frac{\Delta_-(t)}{\Delta_+(t)}$ if $|\mu| < 1$ and Ind $K =$ ind $\Delta(t)$ if $|\mu| > 1$.

Remark 4.4. If, instead of the conditions (4.10) we require that $\left(\frac{t-t_1}{t-\alpha(t_1)}\right)^\nu b(t)$, $\left(\frac{t-t_1}{t-\alpha(t_1)}\right)^\nu d(t) \in C(\Gamma)$, then Theorem 4.3 holds without the assumptions (4.11).

Acknowledgements. This work was partially supported by FCT under the grant PRAXIS/2/2.1/MAT/441/94 and by FCT under the grant PRAXIS XXI/BCC/18977/98 during the first author's visit to Centro de Matemática Aplicada of Instituto Superior Técnico in Lisbon.

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Nikolai K.Karapetiants
Rostov State University, Math. Department
ul. Zorge, 5, Rostov – na – Donu
344104, *Russia*
e – mail : nkarapet@ns.math.rsu.ru

Stefan G.Samko
Universidade do Algarve
Unidade de Ciencias Exactas e Humanas
Campus de Gambelas, Faro, 8000, *Portugal*
e – mail : ssamko@ualg.pt