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APPROXIMATIVE APPROACH TO FRACTIONAL POWERS OF OPERATORS

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Abstract

A new formula is obtained for fractional powers $(-A)^\alpha$ *of operators in a Banach space (which are generators of strongly continuous uniformly bounded semigroups* T_t*). This formula is based on the so called approximative approach and represents the fractional power* $(-A)^\alpha f$ *as a limit of "nice" operators of the form* $\int_0^\infty u_\varepsilon(t)T_t f dt$ *with the elementary function* $u_\varepsilon(t) = \frac{d}{dt}\left[\frac{t}{(t+i\varepsilon)^{1+\alpha}}\right]$.

Key Words and Phrases: fractional powers of operators, semigroups of operators, infinitesimal operator, Balakrishnan formula, fractional differentiation, Marchaud formula, approximative approach

1 Introduction

The well known Balakrishnan formula (see, for instance, [8])

$$(-A)^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (T_t - I) f \frac{dt}{t^{1+\alpha}}\ , \quad 0 < \alpha < 1, \tag{1.1}$$

for the fractional powers $(-A)^\alpha$, where A is the generator of a strongly continuous semigroup in a Banach space X, has a generalization for $\alpha > 1$ in

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the form

$$(-A)^\alpha f = \frac{1}{\varkappa(\alpha,\ell)} \int_0^\infty (T_t - I)^\ell f \frac{dt}{t^{1+\alpha}} \ , \quad \alpha > 0, \tag{1.2}$$

where ℓ is any integer greater than α and $\varkappa(\alpha,\ell) = \int_0^\infty \frac{(1-e^{-t})^\ell}{t^{1+\alpha}} dt$ is the normalizing constant known in fractional calculus (see [7], p. 119), the limit in (1.1)-(1.2) being treated in the norm of X, $f \in D(-A)^\alpha$.

We suggest a new formula for fractional powers of operators which do not use "finite differences" $(T_t - I)^\ell$, based on the idea of approximative approach (we refer for this approach in application to potential type operators to [1],[2],[4],[5] and Section 11 in [6]). In the case $0 < \alpha < 1$ this formula is the following.

$$(-A)^\alpha f = \frac{1}{\Gamma(1-\alpha)} \lim_{\varepsilon \to 0} \int_0^\infty \frac{d}{dt} \left[\frac{t}{(t+i\varepsilon)^{1+\alpha}} \right] T_t f dt. \tag{1.3}$$

This formula is inspired by the approach to fractional derivatives developed in [4]-[5] and [3]. The generalization of this formula for $\alpha > 1$ has the form

$$(-A)^\alpha f = \frac{1}{\Gamma(\ell-\alpha)} \lim_{\varepsilon \to 0} \int_0^\infty \frac{d^\ell}{dt^\ell} \left[\frac{t^\ell}{(t+i\varepsilon)^{1+\alpha}} \right] T_t f dt, \tag{1.4}$$

$$\ell = [\alpha] + 1, \alpha \neq 1, 2, 3, ...$$

2 Preliminaries

1. Let X be a Banach space and T_t a strongly continuous semigroup of linear operators in X. Let

$$Uf = \int_0^\infty u(t) T_t f dt \tag{2.1}$$

where we assume that

$$\int_0^\infty |u(t)| \cdot \|T_t\| dt < \infty. \tag{2.2}$$

The function $u(t)$ will be called the kernel of the operator U.

Lemma 2.1. *Let U and V be two operators of the form (2.1) with kernels $u(t)$ and $v(t)$ satisfying the condition (2.2). Then their composition UV has the same form*

$$UVf = \int_0^\infty w(t) T_t f dt$$

with the kernel $w(t) = \int_0^t u(s)v(t-s) ds$ satisfying the condition (2.2).

The proof is direct with the identity $\|T_{t+s}\| \leq \|T_t\| \cdot \|T_s\|$ taken into account.

2. By $\mathcal{W}_0(R^1)$ we denote the Wiener algebra of Fourier transforms of functions in $L_1(R^1)$. The notation $\mathcal{W}(R^1)$ will stand for the completion of

$\mathcal{W}_0(R^1)$ by the unity. As is well known, for $f \in \mathcal{W}(R^1)$, we have $\frac{1}{f} \in \mathcal{W}(R^1)$ as well, if $f(\xi) \neq 0$ for all $\xi \in \dot{R}^1$, where $\dot{R}^1$ is the compactification of R^1 by the unique infinite point (N.Wiener's theorem).

3. We shall use the function

$$\widehat{\mathcal{K}}^+_{\ell,\alpha}(\xi) = \frac{1}{\varkappa(\alpha,\ell)} \frac{1}{(-i\xi)^\alpha} \int_1^\infty \frac{\left(1-e^{it\xi}\right)^\ell}{t^{1+\alpha}} dt, \quad \ell > \alpha > 0, \tag{2.3}$$

which is known in the fractional calculus, see [7]. It is known that

$$\widehat{\mathcal{K}}^+_{\ell,\alpha}(\xi) \in \mathcal{W}_0(R^1), \quad \ell > \alpha > 0. \tag{2.4}$$

3 Connection between the "truncated" fractional power and the approximating operator

Let

$$(-A)^\alpha_\varepsilon f = \frac{1}{\varkappa(\alpha,\ell)} \int_\varepsilon^\infty (T_t - I)^\ell f \frac{dt}{t^{1+\alpha}} \ , \quad \alpha > 0, \tag{3.1}$$

be the truncated integral in (1.2) and

$$K_\varepsilon f = \frac{1}{\Gamma(\ell-\alpha)} \int_0^\infty \frac{d^\ell}{dt^\ell} \left[\frac{t^\ell}{(t+i\varepsilon)^{1+\alpha}} \right] T_t f dt, \tag{3.2}$$

the approximation from (1.4). We observe that it has the form

$$K_\varepsilon f = \frac{1}{\varepsilon^{1+\alpha}} \int_0^\infty q_\alpha \left(\frac{t}{\varepsilon}\right) T_t f dt \tag{3.3}$$

where

$$q_\alpha(t) = \frac{1}{\Gamma(\ell-\alpha)} \frac{d^\ell}{dt^\ell} \left[\frac{t^\ell}{(t+i)^{1+\alpha}} \right]. \tag{3.4}$$

The operator of the type (2.1)

$$U_\varepsilon f = \frac{1}{\varepsilon} \int_0^\infty u \left(\frac{t}{\varepsilon}\right) T_t f dt \tag{3.5}$$

with $u(t) \in L_1(R^1_+)$ will be referred to as *an identity approximation* in the case when $\int_0^\infty u(t)dt = 1$. Under this condition, $\|U_\varepsilon f - f\| \to 0$ as $\varepsilon \to 0$, if the semigroup T_t is uniformly bounded.

We are interested in the existence of a direct relation

$$K_\varepsilon f = U_\varepsilon (-A)^\alpha_\varepsilon f \tag{3.6}$$

between the truncation (3.1) and the approximation (3.2) via some identity approximation operator U_ε.

In Lemma 3.1 below we denote

$$\lambda = \frac{1}{\alpha\varkappa(\alpha,\ell)} \quad \text{and} \quad a(t) = \frac{1}{\varkappa(\alpha,\ell)t^{1+\alpha}}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}k^\alpha\theta_+(t-k) \in L_1(R^1_+) \tag{3.7}$$

where $\theta_+(t)$ is the Heaviside step function, and use the function $q_\alpha(t)$ defined in (3.4).

Lemma 3.1. *Let $\alpha > 0$ and T_t a strongly continuous and uniformly bounded semigroup. The existence of relation (3.6) is equivalent to existence of a solution $u(t)$ of the Volterra integral equation*

$$\lambda u(t) + \int_0^t a(t-s)u(s)ds = q_\alpha(t), \quad t > 0, \tag{3.8}$$

satisfying the conditions $u(t) \in L_1(R^1_+)$ and $\int_0^\infty u(t)dt = 1$.

Proof. We have

$$(-A)^\alpha \varepsilon f = \frac{1}{\varkappa(\alpha,\ell)}\int_\varepsilon^\infty \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}T_{kt}f\frac{dt}{t^{1+\alpha}}$$

$$= \frac{1}{\varkappa(\alpha,\ell)}\left\{\frac{f(x)}{\alpha\varepsilon^\alpha} + \sum_{k=1}^{\ell}(-1)^k\binom{\ell}{k}\int_{k\varepsilon}^\infty T_{kt}f\frac{dt}{t^{1+\alpha}}\right\}.$$

Hence

$$(-A)^\alpha_\varepsilon f = \frac{1}{\varepsilon^\alpha}\left[\lambda f + \frac{1}{\varepsilon}\int_0^\infty a\left(\frac{t}{\varepsilon}\right)T_t f\, dt\right]. \tag{3.9}$$

Let $\widehat{a}(\xi) = \int_{-\infty}^\infty e^{i\xi t}a(t)dt$ denote the Fourier transformation. For the function $a(t)$ defined in (3.7), an easy calculation yields

$$\widehat{a}(\xi) = \frac{1}{\varkappa(\alpha,\ell)}\int_1^\infty \frac{\left(1-e^{it\xi}\right)^\ell - 1}{t^{1+\alpha}}dt \in \mathcal{W}_0(R^1). \tag{3.10}$$

Consequently,

$$\lambda + \widehat{a}(\xi) = \frac{1}{\varkappa(\alpha,\ell)}\int_1^\infty \frac{\left(1-e^{it\xi}\right)^\ell}{t^{1+\alpha}}dt \in \mathcal{W}(R^1).$$

The right-hand side here occurred in (2.3) so that

$$\lambda + \widehat{a}(\xi) = (-i\xi)^\alpha\widehat{\mathcal{K}}^+_{\ell,\alpha}(\xi) \in \mathcal{W}(R^1). \tag{3.11}$$

To get (3.6), we calculate the composition $U_\varepsilon(-A)^\alpha_\varepsilon$, starting from (3.9). By Lemma 2.1 we obtain

$$U_\varepsilon(-A)^\alpha_\varepsilon f = \frac{1}{\varepsilon^{1+\alpha}}\int_0^\infty b\left(\frac{t}{\varepsilon}\right)T_t f\ dt$$

where

$$b(t) = \lambda u(t) + \int_0^t u(s)a(t-s)ds. \tag{3.12}$$

Then the required relation (3.6) takes the form

$$\frac{1}{\varepsilon^{1+\alpha}} \int_0^\infty q_\alpha\left(\frac{t}{\varepsilon}\right) T_t f \; dt = \frac{1}{\varepsilon^{1+\alpha}} \int_0^\infty b\left(\frac{t}{\varepsilon}\right) T_t f \; dt \tag{3.13}$$

where $q_\alpha(t)$ is the kernel defined in (3.4). To guarantee the validity of (3.13), it suffices to put $q_\alpha(t) = b(t)$, which is nothing else but the Volterra integral equation (3.8). □

Obviously, the solution of the equation (3.8) is given formally via Fourier transforms:

$$\widehat{u}^+(\xi) = \frac{\widehat{q}_\alpha^+(\xi)}{(-i\xi)^\alpha \widehat{\mathcal{K}}_{\ell,\alpha}^+(\xi)} \tag{3.14}$$

where

$$\widehat{u}^+(\xi) = \int_0^\infty u(t)e^{i\xi t}dt \quad \text{and} \quad \widehat{q}_\alpha^+(\xi) = \int_0^\infty q_\alpha(t)e^{i\xi t}dt$$

are limiting values of functions analytic in the upper half plane. We need the solution $u(t)$ supported on R^1_+ so that we have to verify that the ratio in (3.14) is a limiting value of a function analytic in the upper half plane. So we need to check that

$$(-iz)^\alpha \widehat{\mathcal{K}}_{\ell,\alpha}^+(z) \neq 0, \quad \Im z \geq 0. \tag{3.15}$$

Lemma 3.2. *Let* $0 < \alpha < 1, \ell = 1$. *Then*

$$\frac{\widehat{q}_\alpha^+(\xi)}{(-i\xi)^\alpha \widehat{\mathcal{K}}_{\ell,\alpha}^+(\xi)} \in \mathcal{W}_0. \tag{3.16}$$

and the condition (3.15) is satisfied.

Proof. Evidently, for $z = x + iy$ with $y > 0$ we have

$$(-iz)^\alpha \widehat{\mathcal{K}}_{\ell,\alpha}^+(z) = \frac{1}{\varkappa(\alpha,1)} \int_1^\infty \frac{1 - e^{-yt}\cos xt + ie^{-yt}\sin xt}{t^{1+\alpha}} dt \tag{3.17}$$

according to (2.3). Here $1 - e^{-yt}\cos xt \geq 0$, so that the condition (3.15) is satisfied.

To prove (3.16), we use the known fact that

$$\frac{\widehat{q}_\alpha(\xi)}{(-i\xi)^\alpha} \in \mathcal{W}_0, \tag{3.18}$$

see [3], formulas (5.1)-(5.3). Denoting $\widehat{k}(\xi) = \frac{\widehat{q}_\alpha(\xi)}{(-i\xi)^\alpha}$, we have

$$\frac{\widehat{q}_\alpha^+(\xi)}{(-i\xi)^\alpha \widehat{\mathcal{K}}_{\ell,\alpha}^+(\xi)} = \frac{\widehat{q}_\alpha^+(\xi) + \widehat{k}(\xi)}{V(\xi)} \tag{3.19}$$

where

$$V(\xi) = [1 + (-i\xi)^\alpha]\, \widehat{\mathcal{K}}^+_{\ell,\alpha}(\xi).$$

In (3.19) the numerator is in $\mathcal{W}_0$. The same is true for the denominator, by (2.3) and (2.4). By the $\frac{1}{f}$-theorem of N.Wiener, it remains to show that $V(\xi) \neq 0$ for all $\xi \in \dot{R}^1$. For $\xi \in \dot{R}^1 \backslash \{0\}$ this follows from the fact that

$$\Re\alpha \left[(-i\xi)^\alpha \widehat{\mathcal{K}}^+_{\ell,\alpha}(\xi)\right] = \frac{\alpha}{\Gamma(1-\alpha)} \int_1^\infty \frac{1-\cos\xi t}{t^{1+\alpha}} dt \neq 0.$$

For $\xi = 0$ we have $V(0) = \widehat{\mathcal{K}}^+_{\ell,\alpha}(0) = 1$, see [6], p.125. □

4 The main result.

Theorem 4.1. *Let $0 < \alpha < 1$ and T_t a strongly continuous and uniformly bounded semigroup. Then there exists an identity approximation operator U_ε providing the relation (3.6).*

Proof. The proof is prepared by Lemmas 3.1 and 3.2. By Lemma 3.1, the relation (3.6) is reduced to Volterra integral equation (3.8). Since $0 < \alpha < 1$, we may take $\ell = 1$ in (3.7) and then Lemma 3.2 is applicable which provides solvability of the equation (3.8) in $L_1(R^1_+)$. This solution is

$$u(t) = \frac{1}{2\pi} \int_{R^1} e^{-i\xi t} \widehat{u}(\xi) d\xi, \quad \text{where} \quad \widehat{u}(\xi) = \frac{\widehat{q}^+_\alpha(\xi)}{(-i\xi)^\alpha \widehat{\mathcal{K}}^+_{\ell,\alpha}(\xi)}. \tag{4.1}$$

It remains to check the condition $\int_0^\infty u(t)dt = 1$ of Lemma 3.1, that is $\widehat{u}(0) = 1$. To this end, we observe that, besides the property (3.18), it is also known that $\widehat{k}(0) = 1$, $\widehat{k}(\xi) = \frac{\widehat{q}_\alpha(\xi)}{(-i\xi)^\alpha}$, see [3], formula (5.2) and Definition 3.2 there. Then the $\widehat{u}(0) = 1$ follows from another known property $\widehat{\mathcal{K}}_{1,\alpha}(0) = 1$, see [6], p. 125. □

Theorem 4.2. *Let $0 < \alpha < 1$ and T_t a strongly continuous and uniformly bounded semigroup. If $f \in D((-A)^\alpha)$, where $(-A)^\alpha$ is understood as the limit $(-A)^\alpha f = \lim_{\substack{\varepsilon\to 0 \\ (X)}} (-A)^\alpha_\varepsilon f$ of the truncations (3.1), then the limit (1.3) exists as well and coincides with (1.1).*

Proof. By Theorem 4.1, we have the connection (3.6) with the identity approximation U_ε operator, so that $\lim_{\substack{\varepsilon\to 0 \\ (X)}}$ exists and coincides with $g \in X$ for any $g \in X$, whence the statement of the theorem follows, if we take into account that the operators U_ε are uniformly bounded. □

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