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Stefan Samko
Universidade do Algarve, Portugal
UCEH, Campus de Gambelas, Faro, 8000

FRACTIONAL POWERS OF OPERATORS VIA HYPERSINGULAR INTEGRALS

Introduction

The well known Balakrishnan formula represents the fractional power $(-A)^\alpha$ in case of the generator A of a semigroup $T_t, t > 0$, in terms of a (hyper)-singular integral with respect to the variable $t \in R^1_+$, that is,

$$(-A)^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} (T_t - I) f dt \ ,$$

where $0 < \alpha < 1, \ \varphi \in D(A)$, and I is the identity operator. In the case $\alpha > 1$, this formula can be written with the usage of "finite differences" $(T_t - I)^\ell$, $\ell = 1, 2, 3, ..., \ell > \alpha$:

$$(-A)^\alpha f = \frac{1}{\varkappa(\alpha, \ell)} \int_0^\infty t^{-\alpha-1} (I - T_t)^\ell f dt \ , \ell > \alpha, \tag{1}$$

with $\varkappa(\alpha, \ell) = -\Gamma(-\alpha) A_\alpha(\ell)$, where $A_\alpha(\ell) = \sum_{k=0}^{\ell} (-1)^{k-1} \binom{\ell}{k}$. In particular, the fractional power of the Laplace operator is given by (1) with $T_t = P_t$ where P_t is the Poisson semigroup of operators:

$$P_t f = c_n \int_{R^n} \frac{t f(x-y)}{(|x|^2 + t^2)^{(n+1)/2}} dy \ , \quad t > 0.$$

On the other hand, positive fractional powers of the Laplace operator can be given also in the form

$$(-\Delta)^{\frac{\alpha}{2}} f = \frac{1}{d_{n,\ell}(\alpha)} \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}} dy \ , \tag{2}$$

see [23], p.56, which is also known as the Riesz fractional derivative and denoted as $\mathbb{D}^\alpha f = (-\Delta)^{\frac{\alpha}{2}} f$. Here $d_{n,\ell}(\alpha)$ is the known normalizing constant and the finite difference $(\Delta_y^\ell f)(x)$, generated by the standard shift $\tau_y f = f(x-y)$, may be centered one and then $0 < \alpha < \ell$, see [15], or a non-centered and then $0 < \alpha < 2\left[\frac{\ell}{2}\right]$, where $\left[\frac{\alpha}{2}\right]$ stands for the entire part of $\frac{\alpha}{2}$, see [23], Ch.3, Section 1.

Hypersingular constructions of the type (2) can be used for an effective realization of fractional powers of some differential operators of mathematical physics, such as fractional powers $(I-\Delta)^{\frac{\alpha}{2}}$, Δ being the Laplace operator; fractional powers $(-\Delta_x + \frac{\partial}{\partial t})^{\frac{\alpha}{2}}$ of *parabolic (heat) operator* or $(I - \Delta_x + \frac{\partial}{\partial t})^{\frac{\alpha}{2}}$, the Laplace operator being applied in the spatial variable $x = (x_1, \cdots, x_n)$; fractional powers of the *wave operator*, of *Schrödinger operator*and others.

What follows is a brief survey of some of the results on application of hypersingular integrals to the realization of fractional powers of these and other differential operators in partial derivatives. Details of some of the presented results, as well as further applications, may be found in the book [23].

1. The fractional powers $(I-\Delta)^{\frac{\alpha}{2}}$.

According to the Balakrishnan formula, the fractional power $(I-\Delta)^{\frac{\alpha}{2}} f$ may be represented as

$$(I-\Delta)^{\frac{\alpha}{2}} f \ = \frac{1}{\varkappa(\alpha,\ell)} \int_0^\infty t^{-1-\alpha} (I - e^{-t} P_t)^\ell f dt \ , \quad \ell > \alpha \ .$$

Keeping applications in mind, we wish to give a construction of this fractional power directly in terms of multidimensional hypersingular integrals.

a) The idea of the construction. We start with the negative power $(I-\Delta)^{-\frac{\alpha}{2}}$, which is the Bessel potential represented by the convolution

$$(I-\Delta)^{-\frac{\alpha}{2}} \varphi = B^\alpha \varphi \ = \ \frac{\omega_\alpha(|x|)}{|x|^{n-\alpha}} * \varphi, \qquad \alpha > 0, \tag{3}$$

where

$$\omega_\alpha(|x|) = \frac{2^{1-\frac{n+\alpha}{2}}}{\pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} \ |x|^{\frac{n-\alpha}{2}} K_{\frac{n-\alpha}{2}}(|x|) \ = \ \frac{1}{\pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2})} \int_0^\infty t^{\frac{n-\alpha}{2}-1} e^{-t-\frac{|x|^2}{4t}} dt.$$

Since the operators B^α form a semigroup, it is natural to expect that the inverse operator should be formally given by

$$(B^\alpha)^{-1} f \ = \ \frac{\omega_{-\alpha}(|x|)}{|x|^{n+\alpha}} * f \ , \tag{4}$$

under the appropriate interpretation of this convolution.

We note that the function $\omega_{-\alpha}(|x|)$ is differentiable, exponentially decays at infinity and stabilizes at the origin to $\omega_{-\alpha}(0) = \frac{2^{\alpha-1}\Gamma(\frac{n+\alpha}{4})}{\pi^{\frac{n}{2}}\Gamma(-\frac{\alpha}{2})}$. We also observe that formally $\omega_{-\alpha}(|x|) \equiv 0$ for $\alpha = 2, 4, 6, \cdots$, but $\frac{1}{|x|^{n+\alpha}}$, as a distribution, has poles at the same points $\alpha = 2, 4, 6, \cdots$. We describe an effective realization of the convolution (4) with the distribution $\frac{\omega_{-\alpha}(|x|)}{|x|^{n+\alpha}}$.

b) The operator $(I - \Delta)^{\frac{\alpha}{2}}$ as the convolution (4).

Theorem 1. *Let $f(x) \in \mathcal{S}(R^n)$ and $\Re\alpha > 0$. Then*

$$f.p.\frac{\omega_{-\alpha}(|x|)}{|x|^{n+\alpha}} * f = \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{\alpha/2}{k} \Delta^k f + \int_{R^n} \frac{f(x-y) - T_{m-1}(x,y)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|) dy \,, \tag{5}$$

where $m > \Re\alpha$ and $T_m(x,y) = \sum_{|j| \le m} \frac{(-y)^j}{j!}(D^j f)(x)$ is the Taylor polynomial.

We refer to [23], p.274, for the proof of Theorem 1,

Basing on (3), we may write

$$(I - \Delta)^{\frac{\alpha}{2}} f = \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{\alpha/2}{k} \Delta^k f + \int_{R^n} \frac{f(x-y) - T_{m-1}(x,y)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|) dy \,, \tag{6}$$

where $m > \alpha$. We remind that $\omega_{-\alpha}(|y|) \equiv 0$ in the case when $\alpha = 2, 4, 6, \cdots$, so that in this case the right-hand side of (6) turns into just an integer power of $I - \Delta$. Now, having obtained the exact expression for the fractional power, we may justify it directly, firstly for nice function again.

c) Direct justification of the formula (6) in case of nice functions. Let $B^\alpha \varphi = F^{-1}(1 + |\xi|^2)^{-\frac{\alpha}{2}} F\varphi$, $\alpha > 0$, be the Bessel potential operator. A justification of the formula (6) may be given in the following form.

Theorem 2. *Let $f(x) \in \mathcal{S}(R^n)$. Then*

$$(I - \Delta)^{\frac{\alpha}{2}} B^\alpha f = B^\alpha (I - \Delta)^{\frac{\alpha}{2}} f = f \,, \quad \alpha > 0 \,.$$

We consider specially the case $0 < \alpha < 2$. Let $0 < \alpha < 1$ first. The formula (6) turns into

$$(I - \Delta)^{\frac{\alpha}{2}} f := f(x) + \int_{R^n} \frac{f(x-y) - f(x)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|) dy \,. \tag{7}$$

The integral here converges absolutely, for example, for $f \in \mathcal{S}(R^n)$ if $0 < \alpha < 1$. It converges as the limit of truncated integrals over $\{y \in R^n : |y| > \varepsilon\}$ for all $0 < \alpha < 2$. Indeed, since $\omega_{-\alpha}(|y|)$ is even, this follows from the relation:

$$\lim_{\epsilon \to 0} \int_{|y| > \epsilon} \frac{f(x-y) - f(x)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|) dy$$

$$= -\int_{R^n} \frac{(\Delta_y^2 f)(x)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|)dy - \frac{1}{2}\int_{R^n} \frac{(\Delta_y^2 f)(x+y)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|)dy \qquad (8)$$

So, we arrive at the following theorem.

Theorem 3. *Let $0 < \alpha < 2$. The fractional power $(I-\Delta)^{\frac{\alpha}{2}} f$, interpreted as (6), or which is the same, as the operator, inverse to the Bessel potential operator B^α within the framework of the space $\mathcal{S}$, has the form,*

$$(I-\Delta)^{\frac{\alpha}{2}} f \ := f(x) \ + \ \int_{R^n} \frac{f(x-y)-f(x)}{|y|^{n+\alpha}} \omega_{-\alpha}(|y|)dy \ , \qquad (9)$$

where the integral converges absolutely in the case $0 < \alpha < 1$ and conditionally in the case $1 \le \alpha < 2$.

Corollary. *The corresponding realization of $(\lambda^2 I-\Delta)^{\frac{\alpha}{2}}$ with $\lambda > 0$ is*

$$(\lambda^2 I-\Delta)^{\frac{\alpha}{2}} f \ := \lambda^\alpha f(x) \ + \ \int_{R^n} \frac{f(x-y)-f(x)}{|y|^{n+\alpha}} \omega_{-\alpha}(\lambda|y|)dy \ , \quad 0<\alpha<2 \ .$$

In the case $\alpha = 1$, we arrive at the following interpretation of the square root $\sqrt{\lambda I - \Delta}$:

$$\sqrt{\lambda^2 I-\Delta} \ f \ = \ \lambda f(x) \ - \frac{2}{(2\pi)^{\frac{n+1}{2}}} \int_{R^n} \frac{f(x-y)-f(x)}{|y|^{\frac{n+1}{2}}} K_{\frac{n+1}{2}}(\lambda|y|)dy \ . \quad (10)$$

This operator is well known under the name of square root Klein-Gordon operator in mathematical aspects of quantum physics, see e.g. [26], [11] and [22].

In particular, in the planar case $n = 2$ we have $K_{\frac{3}{2}}(z) = \sqrt{\frac{\pi}{2}} \frac{z-1}{z^{\frac{3}{2}}} e^{-z}$ and the formula (10) turns into

$$\sqrt{\lambda^2 I-\Delta} \ f \ = \ \lambda f(x) \ - \frac{1}{2\pi\lambda^{3/2}} \int_{R^2} \frac{f(x-y)-f(x)}{|y|^3} (1-\lambda|y|)e^{-\lambda|y|}dy \ .$$

In the case $1 \le \Re\alpha < 2$ we may also use the form (6) which is

$$(I-\Delta)^{\frac{\alpha}{2}} f \ = \ f(x) \ + \ \int_{R^n} \frac{f(x-y)-f(x)+y\cdot grad\ f(x)}{|y|^{\frac{n+1}{2}}} \omega_{-\alpha}(|y|)dy \ . \quad (11)$$

Here the integral is already absolutely convergent (in case of nice functions).

The following lemma presents another version of the construction (11).

Lemma 4. *Let $1 < \Re\alpha < 2$. The construction (11) may be represented in terms of hypersingular integral*

$$(I-\Delta)^{\frac{\alpha}{2}} f \ = \ f(x) \ + \ \int_{R^n} \frac{f(x)-2f(x-y)+f(x-2y)}{|y|^{\frac{n+1}{2}}} \mu_\alpha(|y|)dy \qquad (12)$$

where $\mu_\alpha(r) \ = \ \frac{1}{2}\sum_{k=1}^\infty 2^{(1-\alpha)k} \omega_{-\alpha}(2^k r)$.

We note that $\mu_\alpha(r) \le cr^{\frac{n+\Re\alpha-1}{2}} e^{-2r}$, $r \ge 1$, and $|\mu_\alpha(r) - \mu_\alpha(0)| \le cr^\beta$, $0 < r \le 1$, where $\beta \in (0, \Re\alpha - 1)$, the constant c not depending on r.

d) Justification of the formula within the framework of the spaces L_p; the cases $0 < \Re\alpha < 1$ and $1 < \Re\alpha < 2$. For nice functions $f(x)$ the operator $(I-\Delta)^{\frac{\alpha}{2}}$ was obtained in (7) which implies a non-absolutely convergent integral in the case $1 \le \Re\alpha < 2$, even in the case of "nice" functions $f(x)$. To deal with "not so nice" functions $f(x)$ in the range $B^\alpha(L_p)$, we shall use only absolutely convergent constructions, that is, the construction (7) in the case $0 < \Re\alpha < 1$ and the construction (12) in the case $1 < \Re\alpha < 2$. But on the whole range $B^\alpha(L_p)$ they will not already be absolutely convergent and will be treated as

$$(I-\Delta)^{\frac{\alpha}{2}} f \;=\; f(x) \;+\; \lim_{\substack{\epsilon\to 0\\ (L_p)}} T_\epsilon^\alpha f \tag{13}$$

with $T_\epsilon^\alpha f \;=\; \int_{|y|>\epsilon} \frac{(\Delta_y^\ell f)(x)}{|y|^{n+\alpha}} \mu_\alpha(|y|) dy$, where $\ell = 2$ and $\mu_\alpha(|y|)$ is a function from (12), if $1 < \Re\alpha < 2$, and $\ell = 1$ and $\mu_\alpha(|y|) = \omega_{-\alpha}(|y|)$ if $0 < \Re\alpha < 1$.

Theorem 5. *Let $f(x) = B^\alpha\varphi, \varphi \in L_p(R^n), 1 < p < \infty, 0 < \Re\alpha < 1$ or $1 < \Re\alpha < 2$. Then*

$$(I-\Delta)^{\frac{\alpha}{2}} f \;=\; \varphi \tag{14}$$

with $(I-\Delta)^{\frac{\alpha}{2}} f$ interpreted according to (13).

2. Parabolic (heat) hypersingular integrals

We consider the fractional powers $\left(\frac{\partial}{\partial t} - \Delta_x\right)^{\frac{\alpha}{2}}$ introduced via the corresponding Fourier multipliers. The negative fractional powers are known as parabolic fractional integrals. The positive fractional powers will be realized as hypersingular integrals. We refer to original papers [17] and [18], see also [23], Chapter 9, Section 2.

a) Parabolic fractional potentials. The fractional parabolic potentials $H^\alpha\varphi$ are introduced via Fourier transforms by the relations

$$F\left(H^\alpha\varphi\right) \;=\left(|\xi|^2 - i\tau\right)^{-\frac{\alpha}{2}} \widehat{\varphi}(\xi,\tau), \tag{15}$$

where

$$\widehat{\varphi}(\xi,\tau) = (F\varphi)(\xi,\tau) = \int_{R^{n+1}} e^{i\xi x + i\tau t} dx dt \tag{16}$$

and $(|\xi|^2 - i\tau)^{-\frac{\alpha}{2}} := (|\xi|^4 + \tau^2)^{-\frac{\alpha}{4}} e^{-\frac{i\alpha}{2}\arg(|\xi|^2 - i\tau)}$ under the choice $\arg(|\xi|^2 - i\tau) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

The potentials $H^\alpha\varphi$ which are, in fact, negative fractional powers of parabolic differential operators $\left(\frac{\partial}{\partial t} - \Delta_x\right)^{-\frac{\alpha}{2}}$, were introduced in [13] and [25]. It is known that the convolution operator H^α has the form

$$(H^\alpha\varphi)(x,t) \;=\; \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_{R_+^{n+1}} \tau^{\frac{\alpha}{2}-1} W(y,\tau)\varphi(x-y, t-\tau) dy d\tau \;, \tag{17}$$

where $W(y,\eta) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the Gauss-Weierstrass kernel.

Remark 6. *The fractional parabolic potential $H^\alpha\varphi$ may be interpreted as a result of the one-dimensional fractional integration applied in the time variable t to the Gauss-Weierstrass operator $W_t\varphi$, that is ,*

$$(H^\alpha\varphi)(x,t) \;=\; \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \tau^{\frac{\alpha}{2}-1}(W_\tau\varphi)(x,t-\tau)d\tau \;. \tag{18}$$

These formulas evidently generate the corresponding Balakrishnan formula when we pass to positive fractional powers. But we remind that we are now interested in the realization of positive powers in terms of multidimensional hypersingular type integrals.

b) Positive fractional powers $\left(\frac{\partial}{\partial t} - \Delta_x\right)^{\frac{\alpha}{2}}, \alpha > 0$. These powers may be constructed effectively as some hypersingular integral operators inverse to the parabolic potential operators H^α. They will contain non-standard finite differences which take into account different behaviour of potentials with respect to the space variable $x \in R^n$ and the time variable $t \in R^1$.

To arrive at the idea of the construction of these fractional powers, we apply the Fourier transform F_x in x to both parts of the equality $f(x,t) = (H^\alpha\varphi)(x,t)$. We get

$$I_+^{\frac{\alpha}{2}}\left[e^{\tau|x|^2}(F_x\varphi)(x,\tau)\right] \;=\; e^{t|x|^2}(F_xf)(x,t) \;,$$

where I_+^α is the one-dimensional fractional integration operator applied in the time variable. Inverting the operator $I_+^{\frac{\alpha}{2}}$ according to the well known Marchaud formula for fractional derivatives, we get

$$e^{t|x|^2}(F_x\varphi)(x,t) \;=\; \frac{1}{\varkappa\left(\frac{\alpha}{2},\ell\right)} \int_0^\infty \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}(F_xf)(x,t-k\tau)e^{(t-k\tau)|x|^2}\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}} \;.$$

Multiplying this by $e^{-t|x|^2}$ and then applying the inverse Fourier transform in x, we arrive at

$$\varphi(x,t) = \frac{1}{\varkappa\left(\frac{\alpha}{2},\ell\right)} \int_0^\infty \left[f(x,t) + \sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\int_{R^n} f(x-y,t-k\tau)W(y,k\tau)dy\right]\frac{d\tau}{\tau^{1+\frac{\alpha}{2}}} \;.$$

Hence, after the change of variables $y \to \sqrt{k\tau}y$ and easy transformations, we arrive at the following

Conclusion 7. *The formal solution of the equation $(H^\alpha\varphi)(x,t) \;=\; f(x,t)$ is given by the formula*

$$\varphi(x,t) \;=\; \frac{1}{(4\pi)^{\frac{n}{2}}\varkappa\left(\frac{\alpha}{2},\ell\right)} \int_{R_+^{n+1}} \frac{(\Delta_{y,\tau}^\ell f)(x,t)}{\tau^{1+\frac{\alpha}{2}}} e^{-\frac{|y|^2}{4}}dyd\tau \;=\; \left(\frac{\partial}{\partial t} - \Delta_x\right)^{\frac{\alpha}{2}} f \;, \tag{19}$$

where

$$(\Delta^{\ell}_{y,\tau} f)(x,t) \;=\; \sum_{k=0}^{\ell}(-1)^k \binom{\ell}{k} f(x-y\sqrt{k\tau}, t-k\tau) \;, \quad \ell > \frac{\alpha}{2} \;. \tag{20}$$

In particular, in the case $0<\alpha<2$ *we have*

$$\left(\frac{\partial}{\partial t}-\Delta_x\right)^{\frac{\alpha}{2}} f \;=\; \frac{1}{(4\pi)^{\frac{n}{2}}\Gamma\left(-\frac{\alpha}{2}\right)} \int_{R^{n+1}_{+}} \frac{f(x,t)-f(x-y\sqrt{\tau}, t-\tau)}{\tau^{1+\frac{\alpha}{2}}} e^{-\frac{|y|^2}{4}} dyd\tau. \tag{21}$$

Remark 8. *The constructions (19) and (22) were obtained formally. The question which arise naturally, is whether the integrals in (19) and (22) converge. For* $f\in\mathcal{S}$ *it is not hard to show that they converge absolutely if we take* $\ell>\alpha$*. In the case* $\ell > \frac{\alpha}{2}$ *they converge non absolutely, as the limits of the corresponding truncated integrals:*

$$H^{-\alpha} f \;=\; \lim_{\epsilon\to 0} H^{-\alpha}_{\epsilon} f \tag{22}$$

where

$$H^{-\alpha}_{\epsilon} f \;=\frac{1}{(4\pi)^{\frac{n}{2}}\varkappa\left(\frac{\alpha}{2},\ell\right)} \int_{R^{n+1}_{\epsilon+}} \frac{(\Delta^{\ell}_{y,\tau} f)(x,t)}{\tau^{1+\frac{\alpha}{2}}} e^{-\frac{|y|^2}{4}} dyd\tau \tag{23}$$

and $R^{n+1}_{\epsilon+} = \{(y,\tau) : y\in R^n, \tau>\epsilon\}, \epsilon>0$.

c) Justification of the inversion in case of nice functions. Below Φ is the Lizorkin test function space, invariant with respect to fractional powers of the Laplace operator, see [23], p.39. The following auxiliary statement is a matter of direct verification, from which Theorem 10 follows.

Lemma 9. *Let* $f = H^{\alpha}\varphi,\ \alpha>0$ *where* $\varphi\in\Phi$. *For the truncated construction* $H^{-\alpha}_{\epsilon} f$ *the representation*

$$H^{-\alpha}_{\epsilon} f = \int_{R^n} W(y,\tau)\mathcal{K}^{+}_{\ell,\frac{\alpha}{2}}(\tau)\varphi\left(x-\sqrt{\varepsilon}y, t/\varepsilon\tau)dyd\tau\right) \tag{24}$$

is valid, where $\mathcal{K}^{+}_{\ell,\alpha}(\tau)$ *is some integrable kernel of the type of the identity approximation.*

Theorem 10. *Let* $f = H^{\alpha}\varphi,\ \alpha>0$ *, where* $\varphi\in\Phi$ *. Then*

$$\left(\frac{\partial}{\partial t}-\Delta_x\right)^{\frac{\alpha}{2}} f := \lim_{\epsilon\to 0} H^{-\alpha}_{\epsilon} f = \varphi \;. \tag{25}$$

d) The case of functions in L_p.

Theorem 11. *Let* $0<\alpha<\frac{n+2}{p},\ 1<p<\infty$ *, and* $f = H^{\alpha}\varphi$ *with* $\varphi\in L_p(R^{n+1})$*. Then*

$$\lim_{\substack{\epsilon\to 0\\(L_p)}} H^{-\alpha}_{\epsilon} f \;=\; \varphi \;.$$

e) The spaces of parabolic potentials. Let

$$H^\alpha(L_p) \ = \ \left\{f: f = H^\alpha\varphi, \varphi \in L_p(R^{n+1})\right\}, \ \ 1 \le p < \infty \ , \ \ \alpha > 0.$$

The range $H^\alpha(L_p)$ is understood in the usual sense only in the case $0 < \alpha < \frac{n+2}{p}$. In the case $\alpha \ge \frac{n+2}{p}$ the potential $H^\alpha\varphi$, with $\varphi \in L_p$, is understood as the convolution with the kernel $\frac{\tau_+^{\frac{\alpha}{2}-1}}{\Gamma(\frac{\alpha}{2})}W(y,\tau)$ in the Lizorkin space $\Phi'(R^{n+1})$ of distributions. The space $H^\alpha(L_p)$ is a Banach space with respect to the norm $\|f\|_{H^\alpha(L_p)} = \|\varphi\|_p$.

Theorem 12 below provides a characterization of the range $H^\alpha(L_p)$. Naturally, this range can be described in terms of convergence of the hypersingular integral $H^{-\alpha}f$. But the heat operator is a quasihomogeneous operator with a non-degenerate symbol. Therefore, it is also natural to expect that the range $H^\alpha(L_p)$ may be also characterized in terms of convergence of the anisotropic hypersingular integral. The corresponding anisotropic distance may be obtained in terms of elementary functions in this case:

$$\rho = \rho(y,\eta) = \left(\frac{\sqrt{|y|^4 + 4|\eta|^2} + |y|^2}{2}\right)^{\frac{n+2}{2(n+1)}} .$$

Let

$$(T^\alpha f)(x,t) = \lim_{\substack{\epsilon\to 0 \\ (L_p(R^{n+1}))}} \int_{\rho(y,\eta)>\epsilon} \frac{(\Delta^{2\ell}_{y,\eta}f)(x,t)}{[\rho(y,\eta)]^{n+\frac{\alpha(n+1)}{n+2}}} dyd\eta \ , \ \ \ell > \frac{\alpha}{2} \ .$$

Theorem 12. *Let* $0 < \alpha < \frac{n+2}{p}$, $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n+2}$. *Then*

$$H^\alpha(L_p) = \{f \in L_q : \ H^{-\alpha}f \in L_p\} = \{f \in L_q : \ T^\alpha f \in L_p\} \ . \tag{26}$$

For the proof of this theorem we refer to [17] for the first of the equalities in (26) and [10] for the second one.

3. Fractional powers of the Schrödinger operator

The results shown below represent a slight modification of what was obtained in [20]. The Schrödinger operator $S = \Delta_x + i\frac{\partial}{\partial t}$, $x \in R^n$, $t \in R^1$ has the symbol $\tau - |\xi|^2$, so that its fractional power may be defined via Fourier transforms as

$$S^{\frac{\alpha}{2}}\varphi = F^{-1}\left[(\tau - |\xi|^2)^{\frac{\alpha}{2}}(F\varphi)(\xi,\tau)\right] \ , \tag{27}$$

at least, formally. We should make sense to the construction (27).

a) Fractional Schrödinger potential operator. Negative fractional powers are realized as integral operators

$$I^\alpha_{Schr}\varphi = S^{-\frac{\alpha}{2}}\varphi = \int_{R^{n+1}_+} s_\alpha(y,\eta)\varphi(x-y, t-\eta)dyd\eta \ , \tag{28}$$

where $R^{n+1}_+ = \{(y,\eta) \in R^{n+1} : y \in R^n, \eta \in R^1_+\}$ and

$$s_\alpha(y,\eta) = \frac{e^{-\frac{\pi i(n+\alpha)}{4}}}{(4\pi)^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}\eta^{\frac{\alpha-n-2}{2}}e^{i\frac{|y|^2}{4\eta}} \ , \quad \widehat{s}_\alpha(\xi,\tau) = (\tau - |\xi|^2 + i0)^{-\frac{\alpha}{2}} \ . \qquad (29)$$

The integral (28) may be referred to as *the Schrödinger fractional potential.* The integral in (64) exists in the usual sense only when $\Re\alpha > n$. If $\Re\alpha \le n$, the kernel $s_\alpha(y,\eta)$ has a non integrable singularity in η at $\eta = 0$. We shall treat the Shrödinger fractional potential with $0 < \Re\alpha \le n$ as analytical continuation in α. But before that we wish to justify the passage of the type (27), even if in the case of nice functions φ.

Let

$$P = \{(\xi,\tau) \in R^{n+1} : \ \xi \in R^n, \tau \in R^1_+, \tau = |\xi|^2\}$$

be the paraboloid surface at which the symbol $\tau - |\xi|^2$ of the Schrödinger operator vanishes, and let Φ_P be the Lizorkin test function space generated by the set P, so that it is invariant with respect to the operator I^α_{Schr}.

For functions in Φ_P the passage to Fourier transforms may be justified. Namely, the following lemma is valid.

Lemma 13. *Let $\Re\alpha > n, \varphi \in \Phi_P$. Then*

$$(FI^\alpha_{Schr}\varphi)(\xi,\tau) = \frac{(F\varphi)(\xi,\tau)}{(\tau - |\xi|^2 + i0)^{\frac{\alpha}{2}}} \ .$$

To deal with the operator I^α_{Schr} in the case $0 < \Re\alpha \le n$, when it does not exist under the direct approach (28), we shall use its analytic continuation which is constructively realized from the half-space $\Re\alpha > n$ into the strip $-2\ell < \Re\alpha \le n$ with an arbitrary $\ell \in \mathbb{N}$ except for some points in this strip, where it may have poles.

b) Analytical continuation of the Schrödinger fractional potential. Let $\varphi(x,t) \in \mathcal{S}(R^{n+1})$. To organize convergence in (28) for $\Re\alpha \le n$ by the familiar usage of finite differences, it is convenient to single out the factor $\frac{1}{\eta^{1-\frac{\alpha}{2}}}$ from $s_\alpha(y,\eta)$, and we introduce the following hypersingular construction

$$I^\alpha_{Schr}\varphi = \lim_{\varepsilon\to 0}\frac{e^{-\frac{\alpha\pi i}{4}}}{d_{n,\ell}(-\alpha)}\int_{R^{n+1}_{+,\varepsilon}}\frac{(\widetilde{\Delta}^\ell_{y,\eta}\varphi)(x,t)}{\eta^{1-\frac{\alpha}{2}}}M(y,\tau)dyd\tau \qquad (30)$$

where $R^{n+1}_{+,\epsilon} = \{(y,\eta) \in R^{n+1} : y \in R^n, \eta > \epsilon\}, \epsilon > 0$,

$$M(y,\eta) = \frac{e^{-\frac{n\pi i}{4}}}{(4\pi\eta)^{\frac{n}{2}}}e^{i\frac{|y|^2}{4\eta}},$$

and $\widetilde{\Delta}^\ell_{y,\eta}\varphi$ is the Marchaud type generalized difference (see [23], p. 78) with different kind of steps in y and η, which reflects the non-homogeneity of the

Schrödinger operator:

$$(\widetilde{\Delta}^{\ell}_{y,\eta}\varphi)(x,t) = \sum_{j=0}^{\ell}(-1)^j C_j^{(\ell)}\varphi(x - a^{\frac{j}{2}}y, t - a^j\eta) \ , \quad \ell \in \mathbb{N} \ ,$$

where $a > 1$ may be chosen arbitrarily. Here the $C_j^{(\ell)}$ are given by the formulas

$$C_i^{(\ell)} = a^{i(\frac{i+1}{2}-\ell)} \frac{\prod_{k=M_i+1}^{\ell}(a^k-1)}{\prod_{k=1}^{m_i}(a^k-1)} \ , i = 1, 2, ..., \ell - 1, \tag{31}$$

$$C_i^{(\ell)} = a^{i(\frac{i+1}{2}-\ell)} \frac{\prod_{k=M_i+1}^{\ell}(a^k-1)}{\prod_{k=1}^{m_i}(a^k-1)} \ , i = 1, 2, ..., \ell - 1, \tag{32}$$

with $m_i = \min(i, \ell - i)$ and $M_i = \max(i, \ell - i)$ and the normalizing constant in (30) is equal to

$$d_{n,\ell}(-\alpha) = \Gamma\left(\frac{\alpha}{2}\right)\sum_{j=0}^{\ell}(-1)^j C_j^{(\ell)} a^{-\frac{\alpha}{2}j} \ .$$

Lemma 14. *Let $\varphi(x,t) \in \mathcal{S}(R^{n+1})$. The limit in (30) exists and represents the analytical continuation of the Schrödinger fractional potential $I^{\alpha}_{Schr}\varphi$ from the half-plane $\Re\alpha > n$ to the half-plane $\Re\alpha > -2\ell$, except, probably, for the points*

$$\alpha = -2m + \frac{4k\pi i}{ln\ a} \ , \quad m = 0, 1, 2, ..., \ell - 1, \quad k \in \mathbb{Z}\backslash\{0\}.$$

c) Positive fractional powers of the Schrödinger operator. Lemma 14 already represents such positive powers since $\Re\alpha$ may be negative in the construction (30). So we define it as

$$\left(\Delta_x + i\frac{\partial}{\partial t}\right)^{\frac{\alpha}{2}} f = J^{-\alpha}_{Schr} f$$

and rewrite this in the convenient way, assuming that $\ell > \frac{\Re\alpha}{2}$:

$$\left(\Delta_x + i\frac{\partial}{\partial t}\right)^{\frac{\alpha}{2}} f = \frac{e^{\frac{\alpha\pi i}{4}}}{d_{n,\ell}(\alpha)} \int_{R^{n+1}_+} \frac{\left(\widetilde{\Delta}^{\ell}_{y,\eta} f\right)(x,t)}{\eta^{1+\frac{\alpha}{2}}} M(y,\eta) dy d\eta \ , \tag{33}$$

where the finite difference is defined by $a > 1$ such that

$$\alpha \neq 2m + \frac{4k\pi i}{ln\ a} \ , \quad m = 0, 1, 2, ..., \ell - 1, \quad k \in \mathbb{Z}\backslash\{0\}. \tag{34}$$

In particular,

$$\sqrt{\Delta_x + i\frac{\partial}{\partial t}}\ f = \tag{35}$$

$$\frac{1+i}{2\sqrt{2\pi}(\sqrt{a}-1)}\int_{R_+^{n+1}}\frac{f(x-y,t-\eta)-f(x-\sqrt{a}y,t-a\eta)}{\eta^{\frac{3}{2}}}M(y,\eta)dyd\eta\ ,$$

where $a>1$ may be arbitrary.

The following theorem states that the positive fractional power of the Schrödinger operator is the inverse operator to the Schrödinger fractional potential operator, at least on "nice" functions.

Theorem 15. *Let $\varphi\in\Phi_P, \Re\alpha>0$. Then*

$$\left(\Delta_x+i\frac{\partial}{\partial t}\right)^{\frac{\alpha}{2}}I^\alpha_{Schr}\varphi=I^\alpha_{Schr}\left(\Delta_x+i\frac{\partial}{\partial t}\right)^{\frac{\alpha}{2}}\varphi\ =\ \varphi, \tag{36}$$

where $\left(\Delta_x+i\frac{\partial}{\partial t}\right)^{\frac{\alpha}{2}}$ is the operator (33) constructed under the choice of $a>1$, satisfying the condition (34) and $I^\alpha_{Schr}\varphi$ is defined by (27) in the case $\Re\alpha>n$ and treated as its analytical continuation (30) when $0<\Re\alpha\le n$.

5. Fractional powers of some other differential operators

We give some brief indications on investigations of fractional powers of differential operators with homogeneous non-constant coefficients (which thereby are not invariant with respect to the translation operator) or of some non-homogeneous differential operators.

The fractional powers

$$\left[-\sum_{k=1}^{n}\left(x_k\frac{\partial}{\partial x_k}\right)^2\right]^{\frac{\alpha}{2}} \tag{37}$$

were studied in [7] - [10].

The negative powers are realized as the Riesz-type potential operator

$$\mathcal{I}^\alpha\varphi=\frac{1}{\gamma_n(\alpha)}\int_{R_+^n}\varphi(x\circ y)\left|\overrightarrow{ln}\ y\right|^{\alpha-n}dy\ ,$$

where $\overrightarrow{ln}\ y=(ln\ y_1,...,ln\ y_n)$ and $R_+^n=\{y\in R^n:y_1>0,...,y_n>0\}$. The positive fractional powers are realized as the corresponding modification of hypersingular integrals, adjusted to the n-tant R_+^n, but with a non-standard truncation of the hypersingular integral when it is interpreted on "not very nice" functions.

The fractional powers of the type

$$\mathbb{I}^\alpha_\pm\varphi=\left[\pm\sum_{k=0}^{n}x_k^2\frac{\partial^2}{\partial x_k^2}\right]^{-\frac{\alpha}{2}}\ ,\ \Re\alpha>0, \tag{38}$$

were considered in [1] with the realization $\mathbb{I}^\alpha_\pm \varphi = \int_{R^n_+} i_\alpha(t)\varphi\left(\frac{x}{t}\right) dt$, where

$$i^\alpha(t) = \frac{n^{\frac{n-\alpha}{4}}}{2^{n-1}\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)} \left|\overrightarrow{ln}\ t\right|^{\frac{\alpha-n}{4}} K_{\frac{n-\alpha}{4}}\left(\frac{\sqrt{n}}{2}\left|\overrightarrow{ln}\ t\right|\right)\sqrt{t_1...t_n}.$$

Positive powers were realized by means of weighted hypersingular integrals.

Another dilatation-invariant version

$$\left(-|x|^2\Delta\right)^{\frac{\alpha}{2}}$$

was considered in [5] with the corresponding realization as potentials in case of negative powers and in the form of non-standard hypersingular integrals in case of positive powers.

The following fractional powers of the operator with constant coefficients, but non-homogeneous one,

$$\left(-P(D,D) + \sum_{k=0}^{n} a_k \frac{\partial}{\partial x_k}\right)^{\frac{\alpha}{2}} \ ,\ a = (a_1, ..., a_n) \in R^n \ , \tag{39}$$

where $P(x, x)$ is an arbitrary positive quadratic form with real coefficients, were treated in [2] but in case of positive fractional powers, instead of the method of hypersingular integrals, there was used a modification of this method in the form of the limit of approximative inverse operators (AIO).

By the same method in [3] and [4] the complex powers (39) were constructed under a weaker assumption that $P(x, x) - ia \cdot x$ is hypoelliptic; a more general case when $1 \leq rang\ P \leq n - 1$ was also treated there.

The case of complex a_k in (39) has a different nature in comparison with that of real coefficients, because the symbol of the operator $-P(D,D) + a \cdot D$ with complex a degenerates on some ellipsoid if $\Re\alpha = 0$ and on the intersection of the ellipsoid and a hyperplane, if $\Re\alpha = 0$. This more difficult case was treated in [14].

Negative fractional powers $(-\Delta - I)^{-\frac{\alpha}{2}}$ are known as fractional acoustic potentials. They have the form

$$(-\Delta - I)^{-\frac{\alpha}{2}}\varphi = \frac{i\ 2^{-\frac{n+\alpha}{2}}}{\pi^{\frac{\alpha}{2}}} \int_{R^n} \frac{H^{(1)}_{\frac{n-\alpha}{2}}(|y|)}{|y|^{\frac{n-\alpha}{2}}}\varphi(x-y)dy \ ,$$

where $0 < \Re\alpha < n + 1$ and $H^{(1)}_\nu$ is the Hankel function of the first kind. The positive powers can be realized by means of AIO. Fractional powers

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - ... - \frac{\partial^2}{\partial x_n^2}\right)^{\frac{\alpha}{2}} \tag{40}$$

of the wave operator by the method of AIO were constructed in [19].

The powers of the Klein-Gordon-Fock operator

$$\left(m^2 I + \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2}\right)^{\frac{\alpha}{2}} \quad , \quad m > 0, \tag{41}$$

were treated in [20] and the powers

$$\left(m^2 I + \frac{\partial}{\partial x_1} + \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \dots - \frac{\partial^2}{\partial x_n^2}\right)^{\frac{\alpha}{2}} \quad , \quad 0 \le m \le \frac{1}{2} \tag{42}$$

including the case of the telegraph operator ($m = 0$) were dealt with in [16]. The negative fractional power ($\Re\alpha < 0$) in (42) proves to be the following potential type operator

$$(H_m^\alpha \varphi)(x) = c_{n,m}(\alpha) \int\limits_{K_+^+} \frac{e^{-\frac{y_1}{2}} I_{\frac{\alpha-n}{2}}(\lambda r(y))}{r^{\frac{n-\alpha}{2}}(y)} \varphi(x-y) dy,$$

with $r^2(y) = y_1^2 - y_2^2 - \cdots - y_n^2$ and $\lambda = \frac{1}{2} - 2m^2$, $0 \le m < \frac{1}{2}$, and the Bessel function replaced by 1 in the case $m = \frac{1}{2}$, $c_{n,m}(\alpha)$ is some constant. The kernel of this potential has locally the same behaviour as the kernel of the Riesz hyperbolic potential $I_\square^\alpha \varphi$, but exponentially decreases at infinity.

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