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# On Fredholm Properties of a Class of Hankel Operators

## 1. Introduction

We study Fredholmness (= Noetherity) of the operators of the form

$$K_a\varphi \ :\equiv \ \lambda\varphi(x) \ - \ P_+aP_-Q\varphi \ = \ f(x) \ , \ x\in R^1, \tag{1.1}$$

in the space $L_p(R^1,|x|^\gamma),\ 1<p<\infty,\ -1<\gamma<p-1$, where

$$S\varphi=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\varphi(t)}{t-x}dt,\quad P_\pm=\frac{1}{2}(I\pm S),$$

$a=a(x)$ is a piece-wise continuous function and $Q\varphi=\varphi(-x)$. In the case $p=2,\gamma=0$ and $a(\infty)=0$ , these equations are reduced to

$$\lambda\psi(x) \ - \int_0^\infty k(x+t)\psi(t)dt=g(x),\ x>0, \tag{1.2}$$

where $\psi=F^{-1}\varphi, g=F^{-1}f$ and $k=F^{-1}a,\ F^{-1}$ being the inverse Fourier transform. Equations of the form (1.2) arise in applications in diffraction theory. They were treated in $L^2(R^1_+)$ in the paper by F.Teixeira [?]. More general equations including also a Wiener-Hopf term in (1.2) were studied in $L^2(R^1_+)$ in [?], [17], [3], [20], [1],[?]. The papers [4], [2] are also relevant.

The operators (1.1) can be considered as a particular case of operators from the algebra generated by the operators of multiplication by piece-wise continuous functions, the singular operator $S$ and the Carleman shift operator $Q$ [?], [7],[8],[16]. Therefore, they can be covered by the general Gohberg-Krupnik theorem on Fredholmness of operators in this algebra which, as is well known, requires the construction of some matrix symbol, the latter being rather complicated just in the case when coefficients are discontinuous at the fixed points of the shift. On the other hand, such operators in the case when coefficients are continuous at the fixed points, can be easily treated by means of some general abstract theorem given by the authors [11], [12], [13], [14], [15], see also [?] and presented below as Theorem 2.16, which allows to obtain the Fredholmness conditions and a formula for the calculation of the index in effective terms.

We show that in case of the operators (1.1), the essential spectrum in $L_p(R^1, |x|^\gamma)$ (which is the set of those $\lambda$ in(1.1), for which $K_a$ is not Fredholm) is described precisely and in simple terms of the so-called standard $\frac{1+\gamma}{p}$−lemniscates, which are unilateral in case of a jump of $a(x)$ at the origin or infinity and bilateral in case of jumps at other points. The main result is given in Theorem A below and the main tools used are Theorem 2.16, Theorems 2.1-2.3, the Poincaré-Bertrand formula and the compactness Theorem 2.4.

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## 2. Preliminaries

### 2.1. Operators with a homogeneous kernel

We remind some well-known results for the equations

$$K\varphi \ :\equiv \ \lambda\varphi(x) - \int_0^\infty k(x,y)\varphi(y)\ dy = \ f(x)\ , 0 < x < \infty\ , \tag{2.1}$$

with a homogeneous kernel $k(x, y)$ of degree -1 :

$$k(tx, ty) \ = \ t^{-1}k(x, y),\ x, y \in R^1_+\ ,\ t > 0, \tag{2.2}$$

in the space

$$L_p\left(R^1_+, x^\gamma\right) \ = \left\{f(x) : \int_0^\infty |f(x)|^p x^\gamma dx < \infty\right\} \tag{2.3}$$

and for the equation similar to (2.1) but over the whole line $R^1$ or over the interval $[-1, 1]$.

**a) The case of the equation (2.1).** We shall often refer to the assumption

$$\int_0^\infty |k(1, y)| y^{-\frac{1+\gamma}{p}} dy \ < \infty\ . \tag{2.4}$$

**Theorem 2.1.** *[18],[19] Let the kernel $k(x,y)$ satisfy the assumptions (2.2) and (2.4). Then the operator $K$ is Fredholm in $L_p\left(R^1_+, x^\gamma\right)$ if and only if it is invertible, and a necessary and sufficient condition for that is iff $\sigma_K\left(ix+1-\frac{1+\gamma}{p}\right)\neq 0,\ x\in \dot{R}^1$, where*

$$\sigma_K(z) \;=\; \lambda \;-\; \int_0^\infty k(1,y)y^{z-1}dy\ . \tag{2.5}$$

**b) The case of the whole line.** For the equation

$$K\varphi \;:\equiv\; \lambda\varphi(x) - \int_{-\infty}^{\infty} k(x,y)\varphi(y)\ dy = \; f(x)\ ,\ x\in R^1,$$

where

$$k(tx,ty) \;=\; t^{-1}k(x,y)\ , x,y\in R^1\ ,\ t>0, \tag{2.6}$$

we assume that

$$\int_{-\infty}^{\infty} |k(\pm 1,y)|\cdot|y|^{-\frac{1+\gamma}{p}}dy \;<\; \infty\ . \tag{2.7}$$

**Theorem 2.2.** *[18],[19] Let $k(x,y)$ satisfy the assumptions (2.6)-(2.7). Then the operator $K$ is Fredholm in the space $L_p\left(R^1, |x|^\gamma\right)$ if and only if it is invertible, and a necessary and sufficient condition for that is*

$$\det\sigma_K\left(ix+1-\frac{1+\gamma}{p}\right)\neq 0,\quad x\in\dot{R}^1\ , \tag{2.8}$$

*where*

$$\sigma_K(z) \;=\; \begin{pmatrix} \lambda-\mathcal{K}_{++}(z) & -\,\mathcal{K}_{+-}(z) \\ -\,\mathcal{K}_{-+}(z) & \lambda-\mathcal{K}_{--}(z)\end{pmatrix} \tag{2.9}$$

*and*

$$\mathcal{K}_{\pm\pm}(z) = \int_0^\infty k(\pm 1,\pm y)y^{z-1}\,dy\ . \tag{2.10}$$

**c) The case of** $[-1,1]$. For the equation

$$K\varphi \;:\equiv\; \lambda\varphi(x) - \int_{-1}^{1} k(x,y)\varphi(y)\ dy = \; f(x)\ ,\quad |x|\le 1, \tag{2.11}$$

the following theorem is valid.

**Theorem 2.3.** *[18],[19] Let $k(x,y)$ satisfy the assumptions (2.6)-(2.7). Then the operator (2.11) is Fredholm in $L_p([-1,1],|x|^\gamma)$ , iff the condition (2.8) is satisfied and then*

$$Ind\ K = -w\left(\det\sigma_K\right)$$

*where*

$$w\left(\det\sigma_K\right) \;=\frac{1}{2\pi}\arg\ det\ \sigma_K\left(ix+1-\frac{1+\gamma}{p}\right)\Big|_{-\infty}^{\infty}\ ,$$

*is the winding number of the function* $\det \sigma_K$ *.*

**d) Compactness theorem.** Let

$$T\varphi \ :\equiv \ \int_0^\infty \ a(x,y)k(x,y)\varphi(y) \ dy = \ f(x) \ , x > 0, \tag{2.12}$$

where the homogeneous kernel $k(x,y)$ satisfies the assumptions (2.2) and (2.4).

**Theorem 2.4.** *[10] Let* $a(x,y) \in L_\infty(R^2)$ *and* $a(+0,+0) = a(+\infty,+\infty) = 0$ *in the sense that*

$$\lim_{N\to\infty} \ esssup_{0<x<\frac{1}{N}} \ esssup_{0<y<\frac{1}{N}} \ |a(x,y)| = \lim_{N\to\infty} \ esssup_{x>N} \ esssup_{y>N} \ |a(x,y)| \ = \ 0 \ ,$$

*then the operator* $T$ *is compact in* $L_p(R^1_+, |x|^\gamma)$ *.*

**e) The Carleman equation.** The equation

$$K\varphi \ :\equiv \ \lambda\varphi(x) - \frac{1}{\pi}\int_0^\infty \frac{\varphi(y) \ dy}{x+y} \ = \ f(x) \ , \ 0 < x < \infty \ , \tag{2.13}$$

known as the Carleman equation, is immediately covered by Theorem 2.1. In this case the condition (2.4) gives $-1 < \gamma < p - 1$ . We have

$$\sigma_K(z) = \lambda - \frac{1}{\pi}\int_0^\infty \frac{y^{z-1}dy}{1+y} \ = \ \lambda - \frac{1}{\sin \pi z} \ , \quad \Re z > 0 \ ,$$

see [9], 3.222.2 , so that

$$\sigma_K\left(ix + 1 - \frac{1+\gamma}{p}\right) = \lambda - \frac{1}{\sin\left(\frac{1+\gamma}{p} - ix\right)\pi} \ , \quad 0 < \frac{1+\gamma}{p} < 1 \ . \tag{2.14}$$

**Lemma 2.5.** *The range of the function* $g_\alpha(x) = \frac{1}{\sin(\alpha - ix)\pi}$, $x \in R^1$, *with* $0 < \alpha < 1$, *is the interval* $[0,1]$ *in the case* $\alpha = \frac{1}{2}$ *and the lemniscate*

$$E_\alpha \ = \ \left\{ z = re^{i\varphi} : r^2 = \frac{4}{\sin^2 2\alpha\pi}\cos(\varphi + \pi\alpha)\cos(\varphi - \pi\alpha) \right\} \tag{2.15}$$

*in the case* $\alpha \neq \frac{1}{2}$ *, or*

$$\left(\frac{u}{\sin \alpha\pi}\right)^2 \ + \ \left(\frac{v}{\cos \alpha\pi}\right)^2 \ = \ (u^2 + v^2)^2, \ u > 0; \ u + iv = z \tag{2.16}$$

*in cartezian coordinates.*

Proof. The case $\alpha = \frac{1}{2}$ is obvious since $g_{1/2}(x) = \frac{1}{ch\pi x}$. In the general case we have

$$u = \Re g_\alpha(x) = \frac{A\xi}{(A\xi)^2 + (B\eta)^2} > 0 \ , \quad v = \Im g_\alpha(x) = \frac{B\eta}{(A\xi)^2 + (B\eta)^2} \ , \tag{2.17}$$

where $A = \sin \pi\alpha$ , $B = \cos \pi\alpha$, $\xi = ch\pi x$ and $\eta = sh\pi x$. Hence $u^2+v^2 = \frac{1}{(A\xi)^2+(B\eta)^2}$. Then from (2.17) we obtain $\xi = \frac{1}{A}\frac{u}{u^2+v^2}, \eta = \frac{1}{B}\frac{v}{u^2+v^2}$ . Since $\xi^2 - \eta^2 \equiv 1$ , this yields (2.16), which is easily transformed to the equation in polar coordinates in (2.15). □

The direct application of Theorem 2.1 to (2.13) with Lemma 2.5 taken into account gives the following result.

**Theorem 2.6.** *The operator (2.13) is Fredholm (invertible) in $L_p(R^1_+, x^\gamma)$, $-1 < \gamma < p-1$, iff $\lambda \notin E_{\frac{1+\gamma}{p}}$.*

## 2.2. The standard $\alpha$-lemniscate.

In the previous subsection we arrived at the lemniscate $E_\alpha$ given by the equation in polar coordinates

$$r^2 = \frac{4}{\sin^2 2\alpha\pi}\cos(\varphi+\pi\alpha)\cos(\varphi-\pi\alpha) \tag{2.18}$$

which can be also rewritten as

$$\cos^2\varphi = \frac{\sin^2 2\alpha\pi}{4}r^2 + \sin^2\alpha\pi. \tag{2.19}$$

It will play a crucial role in the formulation of our main result. By this reason we give the following definition.

**Definition 2.7.** We call the curve (2.18) the *standard $\alpha$-lemniscate.*
We need some more information about this curve. The following lemmas are valid.

**Lemma 2.8.** *The lemniscate $E_\alpha$, $0 < \alpha < 1$, is symmetrtic with respect to the half-axis $R^1_+$ , has the "vertex" $\left(\frac{1}{\sin\alpha\pi}, 0\right)$ and lies within the sector*

$$-\pi\left|\frac{1}{2}-\alpha\right| \le \varphi \le \pi\left|\frac{1}{2}-\alpha\right|. \tag{2.20}$$

Proof. The symmetry is obvious, while (2.20) is seen from (2.17), because (2.17) implies $\frac{v^2}{u^2} = \left(\frac{B\eta}{A\xi}\right)^2 \le ctg^2\alpha\pi$ , that is, $tg^2\varphi \le tg^2\left(\frac{\pi}{2}-\alpha\pi\right)$. The "vertex" is obtained at $\varphi = 0$, which gives $\max_\varphi r = \frac{1}{\sin\alpha\pi}$. □

**Corollary 2.9.** *The standard lemniscate $E_\beta, 0 < \beta < 1$, lies inside of the leaf bounded by another lemniscate $E_\alpha, 0 < \alpha < 1$, iff*

$$\min(\alpha, 1-\alpha) < \beta < \max(\alpha, 1-\alpha) .$$

Indeed, we observe that the lemniscates $E_\alpha$ and $E_\beta$ , $\alpha \ne \beta$, have no common points except for the origin, which can be easily derived from the equation (2.19). So, it suffices to determine when the "vertex" of $E_\beta$ lies inside of the leaf of $E_\alpha$, that is, $\sin\beta\pi > \sin\alpha\pi$. The latter is equivalent to the above inequalities for $\beta$ .

**Corollary 2.10.** *A point $\lambda \in \mathbf{C}$ lies inside of the leaf of the standard lemniscate $E_\alpha$,*
*$0 < \alpha < 1$, iff*

$$|\arg \lambda| < \pi|\frac{1}{2} - \alpha|$$

*and for such values of $\varphi = \arg \lambda$*

$$|\lambda| < \frac{2\sqrt{\cos(\varphi + \pi\alpha)\cos(\varphi - \pi\alpha)}}{|\sin 2\alpha\pi|} .$$

**Lemma 2.11.** *The lemniscate $E_\alpha$, $0 < \alpha < 1$ , can be represented in the parametric form*

$$z = \ell_\alpha(t) = \frac{2ite^{\alpha\pi}}{e^{2\alpha\pi i} - t^2} = u(t) + iv(t), \quad 0 < t < \infty, \tag{2.21}$$

*where*

$$u(t) = \frac{2t(1+t^2)\sin\alpha\pi}{t^4 - 2t^2\cos 2\alpha\pi + 1}, \qquad v(t) = \frac{2t(1-t^2)\cos\alpha\pi}{t^4 - 2t^2\cos 2\alpha\pi + 1}, \tag{2.22}$$

*the connection between $t$ and $\varphi$ being given by*

$$t^2 = \frac{ctg\alpha\pi - tg\varphi}{ctg\alpha\pi + tg\varphi} . \tag{2.23}$$

Proof. It is easier to derive (2.18) from (2.21) than *vice versa.* The idea of arriving at (2.21) will become clear later, see (5.29). To derive (2.18) from (2.21)-(2.22), we first observe that from (2.22)

$$tg\ \varphi = \frac{v(t)}{u(t)} = \frac{1-t^2}{1+t^2}\ ctg\alpha\pi.$$

Hence the connection (2.23) follows. From (2.21)-(2.22) we also have

$$r^2 = |z|^2 = \frac{4t^2}{t^4 - 2t^2\cos 2\alpha\pi + 1} . \tag{2.24}$$

Substituting (2.23) into (2.24) we arrive at the equation in (2.18) after easy calculations. □

## 2.3. Fredholmness of some "composite" singular operators and their representation as a composition of "usual" singular operators.

Let $\Gamma$ be a closed smooth curve and let

$$S\varphi = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau$$

be the singular operator. Let $P_\pm = \frac{1}{2}(I \pm S)$. We consider the "composite" operator of the special type:

$$M = P_+mP_+nP_+ + P_+rP_+sP_+ + P_-, \tag{2.25}$$

where $m(t), n(t), r(t)$ and $s(t)$ are piece-wise continuous functions on $\Gamma$ . Operators of the form (2.25) and, in general, operators in the algebra generated by $S$ and the operators of multiplication by piece-wise continuous functions were studied by I.Gohberg and N.Krupnik [6], [5], see also [7]-[8]. The criterion of such operators to be Fredholm in $L_p$ or in $L_p$ with the power weight is known to be given in terms of the matrix symbol. In case of operators (2.25) the final result can be given in usual terms of jumps of the argument of some scalar functions. This is given in Theorem 2.12 below and is based on some simple represenation given by Lemma 2.13.

By $t_1, \cdots, t_m$ we denote the discontinuity points of the coefficients $m(t), n(t), r(t)$ and $s(t)$ . We introduce the following notations

$$\Delta(t) \ = \ m(t)n(t) \ + \ r(t)s(t), \tag{2.26}$$

$$h(t) \ = \ m(t-0)n(t+0) \ + \ m(t+0)n(t-0) \ + \ r(t-0)s(t+0) \ + \ r(t+0)s(t-0) \ ,$$

$$\nu_k^{\pm} = \frac{h(t_k) \ \pm \ \sqrt{h^2(t_k) - 4\Delta(t_k-0)\Delta(t_k+0)}}{2\Delta(t_k-0)}, \ k = 1, 2, \cdots, m, \tag{2.27}$$

where any one of two possible values of the square root is chosen.

Let

$$\omega(t) = \prod_{k=1}^{m} (t - z_0)^{-\frac{1}{2\pi i} ln \ \nu_k^+} \ , \tag{2.28}$$

where $z_0$ is any point inside the domain bounded by $\Gamma$ and the choice of a branch for $ln \ \nu_k^+ = ln \ |\nu_k^+| + i \arg \nu_k^+$ will be indicated in Theorem 2.12 below, formula (2.29).

**Theorem 2.12.** *The operator (2.25) is Fredholm in $L_p(\Gamma), \ 1 < p < \infty$, iff*

$$1) \qquad \inf_{t\in\Gamma} |\Delta(t)| \neq 0 \ ,$$

$$2) \qquad \arg \nu_k^+ \neq \frac{2\pi}{p'} \ (mod \ 2\pi) \ , \ \arg \nu_k^- \neq \frac{2\pi}{p'} \ (mod \ 2\pi) \ .$$

*Then under the choice*

$$-\frac{2\pi}{p} \ < \arg \nu_k^+ < \ \frac{2\pi}{p'} \tag{2.29}$$

*the index of the operator (2.25) is calculated as*

$$Ind \ M \ = \ -ind_p \ \frac{\Delta}{\omega} \ , \tag{2.30}$$

*where $\omega(t)$ is the function (2.28).*

Theorem 2.12 is a consequence of the following lemma.

**Lemma 2.13.** *Let $\Delta(t_k \pm 0) \neq 0$. Then the operator $M$ is representable as*

$$M = (P_+ \frac{\Delta}{\omega} P_+ + P_-)(P_+ \omega P_+ + P_-) + T_1 \ = \ (P_+ \omega P_+ + P_-)(P_+ \frac{\Delta}{\omega} P_+ + P_-) \ + T_2 \ ,$$

*where $T_1$ and $T_2$ are compact operators in $L_p(\Gamma)$ and the operator $P_+\omega P_+ + P_-$ is invertible under the choice* $-\frac{2\pi}{p} < \arg\nu_k^+ < \frac{2\pi}{p'}$, $k = 1, \cdots, m$.

Lemma 2.13 and Theorem 2.12 were proved in [13]-[14], see also [?], Section 1.3.

**Remark 2.14.** We note that $ind_p$ in (2.30) stands for the $p$-index [6] which can be calculated by the formula [?]

$$ind_p\ a(t) \ = \ \frac{1}{2\pi}\sum_{k=1}^{m}(\theta_k - \beta_k) \ ,$$

$$\theta_k \ = \ \int_{t_k+0}^{t_{k+1}-0} darg\ a(t)\ ,\ (t_{m+1} = t_1);\ \ \beta_k \ = \ arg\ \frac{a(t_k - 0)}{a(t_k + 0)} \ \in \ \left(-\frac{2\pi}{p'}\ ,\ \frac{2\pi}{p}\right)\ .$$

**Remark 2.15.** Theorem 2.12 is valid also in the case $\Gamma = R^1$ , if $a(x) \in PC(R^1)$ has discontinuities at a finite number of points $x_1, \cdots, x_m,\ |x_j| < \infty,\ j = 1, \cdots, m,$ and $\omega(x)$ in (2.28) is defined as

$$\omega(x) = \prod_{k=1}^{m}\left(\frac{x-i}{x+i}\right)_{x_k}^{-\frac{1}{2\pi i}ln\ \nu_k^+} , \tag{2.31}$$

where $\omega_k(z) \ = \ \left(\frac{z-i}{z+i}\right)_{x_k}^{\beta_k}$ denotes the choice of the branch of the power function defined by the cut joining the points $i$ and $-i$ and passing through the point $x_k$. For real values $z = x$ it may be represented as

$$\omega_k(x) \ = \ \left(\frac{x-i}{x+i}\right)_{x_k}^{\beta_k} \ = \ \theta_k(x)e^{-2i\beta_k\ arctg\frac{1}{x}} \ ,$$

where $\theta_k(x) = 1$ if $x \in [0, x_k]$ and $\theta_k(x) = e^{2\beta_k\pi i}$ if $x \notin [0, x_k]$ and $\theta_k(x) \equiv 1$ in the case $x_k = 0$ . So,

$$\frac{\omega(x_k - 0)}{\omega(x_k + 0)} \ = \ e^{2\beta_k\pi i}$$

and the function $\omega_k(x)$ is $p$-non-singular (in the terminology of [6]) if $\Re\beta_k \neq \frac{1}{p}(mod\ 1)$ and

$$ind_p\ \omega_k(x) \ = \ \left[\Re\beta_k + \frac{1}{p'}\right] \ . \tag{2.32}$$

## 2.4. Some general theorems on Fredholmness of equations with an involutive operator

Let $X$ be a Banach space and $Q$ a bounded linear operator satisfying the relation $Q^2 = I$ , $Q \neq \pm I,$ and thereby called *involutive.* Let $A$ and $B$ be linear bounded operators, acting in $X$ . We denote

$$A_1 \ = \ QAQ\ ,\ B_1 \ = \ QBQ\ .$$

In applications it usually turs out that the operators $A_1$ and $B_1$ "do not contain" the involution $Q$ in the sense that they are of the same nature as the initially considered operators $A$ and $B$ up to a compact additive term.

We say that two operators on $X$ *quasicommute* if their commutator is compact in $X$ .

*Axiom 1.* The operator $A$ quasicommutes with the operators $B, A_1$ and $B_1$.

*Axiom 2.* The operator $A$ may be approximated, in the operator topology, by Fredholm operators $A_\epsilon$ which quasicommute with $B$.

*Axiom 3.* There exists a (bounded linear) Fredholm operator $U$ which quasicommutes with $A$ and $B$ but anti-quasicommutes with $Q$ , that is

$$UQ \ + QU \qquad \textit{is compact}.$$

**Theorem 2.16.** *Let $A$ and $B$ satisfy Axioms 1-3. Then the operator $K = A + QB$ is Fredholm in $X$ iff the operator*

$$M \ = \ AA_1 \ - \ BB_1$$

*is Fredholm in $X$ and*

$$Ind\ K = \frac{1}{2}\ Ind\ M. \tag{2.33}$$

This theorem was proved in [11], [13]-[14], see also its presentation in [**?**]. We give its proof here for completeness, since we slightly modified the system of axioms in comparison with that in [11], [13]-[14] and [**?**].

Proof.

1) We first note that the operators $K = A+QB$ and $K' = A-QB$ are simultaneously Fredholm and have equal indices:

$$Ind\ K \ = \ Ind\ K'\ .$$

Indeed, by Axiom 3 we have $U(A + QB) \ = \ (A - QB)U + T$, where $T$ is a compact operator.

2). *Sufficiency.* Let $\tilde{K} \ = \ A_1 \ - \ QB$. By Axiom 1 we have

$$K\tilde{K} \ = \ AA_1 \ - \ BB_1 + T_1, \tag{2.34}$$

$$\tilde{K}K \ = \ AA_1 \ - \ BB_1 + T_2, \tag{2.35}$$

where $T_1$ and $T_2$ are compact operators. Then Fredholmness of $M$ implies that of $K$.

3) *Formula for the index.* It suffices to show that

$$Ind\ K \ = \ Ind\ \tilde{K}$$

since in this case (2.34) yields (2.33). To this end, we write

$$A_1(A + QB) \ = \ AA_1 + QAB + T_3\ , \tag{2.36}$$

$$(A_1 - QB)A \ = \ AA_1 - QAB + T_4\ . \tag{2.37}$$

Suppose that $A$ is Fredholm. Then $A_1$ is the same. Then the righthand sides in (2.36)-(2.37) are operators of the type $K$ and $K'$ and by the part 1) of the proof with Axiom 1 taken into account, we get $Ind\ (AA_1 \ + QAB) \ = Ind\ (AA_1 \ - QAB)$ and then

$$Ind\ (A + QB) \ = Ind\ (A - OB).$$

If $A$ is not Fredholm, it just suffices to make use of Axiom 2.

4) *Necessity.* Let $K$ be Fredholm. Then by the part 1) of the proof, $K'$ is Fredholm as well. Let $R_K$ and $R_{K'}$ be their regularizers. Obviously

$$\frac{1}{2}(R_K + R_{K'})A \ + \ \frac{1}{2}(R_K - R_{K'})QB \ = \ I \ + \ T_5 \ , \tag{2.38}$$

$$\frac{1}{2}(R_K - R_{K'})A \ + \ \frac{1}{2}(R_K + R_{K'})QB \ = \ T_6 \ . \tag{2.39}$$

Now, we multiply (2.38) first by $\frac{1}{2}Q(R_K - R_{K'})$ from the left and by $B_1$ from the right and then by $\frac{1}{2}(R_K + R_{K'})Q$ from the left and by $QA$ from the right. Similarly we multiply (2.39) first by $\frac{1}{2}(R_K + R_{K'})Q$ from the left and by $-B_1$ from the right and then by $\frac{1}{2}Q(R_K - R_{K'})$ from the left and by $-QA$ from the right. Summing all the four results, we arrive at

$$R(AA_1 \ - \ BB_1) \ = \ \frac{1}{2}\,[fQ(R_K - R_{K'})B_1 \ + \ (R_K + R_{K'})A] \ + \ T_7 \ , \tag{2.40}$$

where

$$R = \frac{1}{4}\left[f(R_K + R_{K'})Q(R_K + R_{K'})Q - Q(R_K - R_{K'})^2Q\right].$$

Multiplying (2.40) by $Q$ both from the left and from the right and summing the results, we obtain after easy calculations

$$(R + QRQ)(AA_1 - BB_1) \ = 2I + T_8 \ .$$

Therefore, the operator $\frac{1}{2}(R + QRQ)$ is the left regularizer of $AA_1 - BB_1$. Similarly it is checked that it is also a right regularizer. Consequently, $AA_1 - BB_1$ is Fredholm. □

**Remark 2.17.** Axiom 2 was used only for the proof of the formula (2.33) for the index.

## 3. The main results and approaches

### 3.1. Formulation of the main result.

We need the following notations

$$s_0 \ = \ a(+0) \ - a(-0) \ , \ s_\infty = \ a(+\infty) \ - a(-\infty),$$

$$(3.1) \qquad s_k = \sqrt{[a(x_k+0) - a(x_k-0)][a(-x_k+0) - a(-x_k-0)]}$$

where $x_k$ are the discontinuity points of the function $a(x) \in PC(R^1)$ . We observe that we follow here the notation from the paper [?], but the notation $s_\infty$ has the opposite sign.

We introduce also the notation

$$E_\alpha(z_0) = z_0 \cdot E_\alpha = \{\zeta \in \mathbf{C} : \zeta = z_0 \cdot w \ , w \in E_\alpha\}, \ z_0 \in \mathbf{C},$$

for the rotated and dilated standard $\alpha$-lemniscate (see Subsection 2.2) with the "vertex" at the point $\frac{z}{\sin \alpha\pi}$. In particular, $E_{\frac{1}{2}}(z_0) = [0, z_0], \ z_0 \in \mathbf{C}$.

Let

$$\Omega_k^{int} = \Omega_{k,+}^{int} \bigcup \Omega_{k,-}^{int} \ , k = 1, \cdots, m,$$

be the union of the interiors of the leafs of two lemniscates:

$$\Omega_{k,\pm}^{int} = interior \ of \ E_{\frac{1}{p}} \left( \pm \frac{is_k}{2} \right) .$$

Similarly by

$$\Omega_0^{int} \text{ and } \mathit{\Omega}_\infty^{int}$$

we denote the interior of the unilateral lemniscates

$$E_{\frac{1+\gamma}{p}} \left( \frac{-is_0}{2} \right) \quad \text{and} \quad E_{\frac{1+\gamma}{p}} \left( \frac{-is_\infty}{2} \right) ,$$

respectively. We remark that the sets $\Omega_k^{int}$and $\Omega_k^{int}, k = 1, \cdots, m$, are empty in the case $p = 2$ , while $\Omega_0^{int}$ and $\Omega_\infty^{int}$ are empty in the case $\gamma = \frac{p}{2} - 1$. By

$$\chi_k(\lambda) = \chi_{\Omega_k^{int}}(\lambda), \quad k = 0, 1, \cdots, m,$$

we denote the characteristic function of the leaf $\Omega_k^{int}$. Similarly, $\chi_\infty(\lambda) = \chi_{\Omega_\infty^{int}}(\lambda)$ .

**Theorem A.** Let $a(x) \in PC(R^1)$. The operator $K_a$ is Fredholm in the space $L_p(R^1, |x|^\gamma), \ \ 1 < p < \infty \ , -1 < \gamma < p - 1$, iff

$$(3.2) \quad \lambda \notin E_{\frac{1+\gamma}{p}} \left( \frac{-is_0}{2} \right) \bigcup E_{\frac{1+\gamma}{p}} \left( \frac{-is_\infty}{2} \right) \bigcup_{j=1}^{m} \left( E_{\frac{1}{p}} \left( \frac{-is_j}{2} \right) \bigcup E_{\frac{1}{p}} \left( \frac{is_j}{2} \right) \right) .$$

Under the condition (3.2)

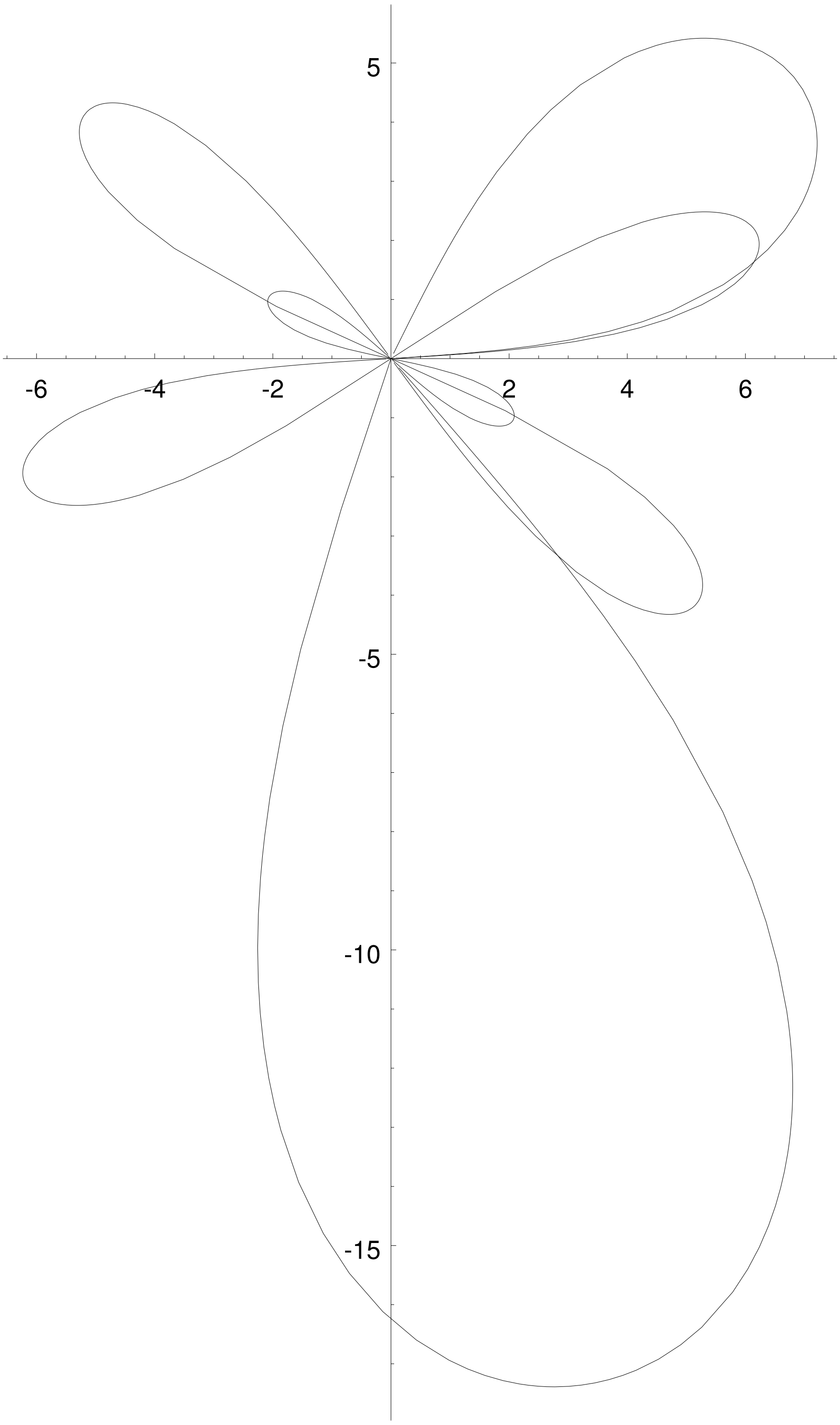
$$(3.3) \quad Ind \ K_a = -sign \ (p-2) \sum_{k=1}^{m} \chi_k(\lambda) + sign(p-2-2\gamma) \left[ \chi_\infty(\lambda) - \chi_0(\lambda) \right] .$$

Theorem A shows that the contribution of the jumps $s_0$ and $s_\infty$ of the function $a(x)$ at the fixed points of the shift into the essential spectrum of the operator $K_a$ consists of the unilateral lemniscates, while the contribution of the jumps at the points $\pm x_j \neq 0$ lead to the bilateral lemniscates, see the figure representing the essential spectrum, where

$$p = \frac{12}{5}, \quad \gamma = \frac{3}{5}, \quad m = 3$$

$$s_1 = 8 + 10i, \quad s_2 = 2 + 4i, \quad s_3 = -4 + 12i$$

$$s_0 = -8 + 12i, \quad s_\infty = 30 - 6i$$


5
-6
-4
-2
2
4
6
-5
-10
-15

**Corollary.** *We have Ind* $K_a = 0$ *if one of the following conditions 1) or 2) is satisfied:*

$$1) \qquad |\lambda| > \max\left\{ \frac{\max_{1\le k\le m}|s_k|}{2\sin\frac{\pi}{p}} \ , \ \frac{\max(|s_0|,|s_\infty|)}{2\sin\frac{\pi(1+\gamma)}{p}} \right\};$$

*or*

$$2) \qquad \left|\arg\left(\frac{i\lambda}{s_k}\right)\right| \ > \ \pi\left|\frac{1}{2}-\frac{1}{p}\right| \ , k=1,\cdots,m, \quad \text{and}$$

$$\left|\arg\left(\frac{i\lambda}{s_0}\right)\right| \ > \ \pi\left|\frac{1}{2}-\frac{1+\gamma}{p}\right|, \quad \left|\left(\arg\frac{i\lambda}{s_\infty}\right)\right| \ > \ \pi\left|\frac{1}{2}-\frac{1+\gamma}{p}\right| \ ,$$

*each of 1) or 2) separately meaning that* $\lambda$ *lies outside of every lemniscate participating in (3.2).*

Theorem A is proved in Section 5. To obtain its corollary, it suffices to refer to Corollary 2.10.

**Remark 3.1.** For simplicity we restricted ourselves to the case of the weight $\rho(x) = |x|^\gamma$. However, the results can be easily extended to the case of the weight function of the form

$$\rho(x) \ = \ |x|^{\gamma_0}\prod_{j=1}^m |x^2-x_j^2|^{\gamma_j}.$$

Instead of (3.2) in this case one gets

$$\lambda \notin \ E_{\frac{1+\gamma_0}{p}}\left(\frac{-is_0}{2}\right) \bigcup E_{\frac{1+\gamma_\infty}{p}}\left(\frac{-is_\infty}{2}\right) \bigcup_{j=1}^m \left( E_{\frac{1+\gamma_j}{p}}\left(\frac{-is_j}{2}\right) \bigcup E_{\frac{1+\gamma_j}{p}}\left(\frac{is_j}{2}\right) \right) \ ,$$

where

$$\gamma_\infty \ = \ \gamma_0 + 2\sum_{j=1}^m \gamma_j \ ,$$

$-1 < \gamma_j < p-1$ , $j = 0,\cdots,m$; $-1 < \gamma_\infty < p-1$ and

$$\mathit{Ind}\ K_a \ = \ -\sum_{k=0}^m \mathit{sign}\ (p-2-2\gamma_k)\chi_k(\lambda) \ + \ \mathit{sign}(p-2-2\gamma_\infty)\chi_\infty(\lambda)$$

with $\chi_k(\lambda), k = 1,\cdots,m$, defined as before, but by the lemniscates $E_{\frac{1+\gamma_k}{p}}\left(\pm\frac{is_k}{2}\right)$ and similarly, by $E_{\frac{1+\gamma_\infty}{p}}\left(\frac{-is_\infty}{2}\right)$ for $\chi_\infty(\lambda)$.

### 3.2. The scheme of the proof and some basic ideas.

**a) The scheme of the proof.** The proof of Theorem A is splitted into the following main steps:

1. We separate the discontinuity points so that it suffices to study separately the operator with a jump of the function $a(x)$ either at the origin, or at infinity or at the pair of symmetric points $\pm x_j \neq 0$ only, see Lemma 3.2 below.
2. The case of a discontinuity at $\pm x_j \neq 0$ is covered by the general Theorem 2.16, see Subsection 5.4.
3. The case of a discontinuity at the origin is treated through the direct calculation of the composition $P_+ a P_- Q$ by means of the Poincaré-Bertrand formula and application of the results on Fredholmness of the operators with homogeneous kernels presented in Subsection 2.1, see Subsection 5.2.
4. The case of a jump at infinity is immediately reduced to the case of a jump at the origin.

Because of applications connected with the case $p = 2, \gamma = 0$ we treat this case separately in Section 4, which yields simpler proofs.

**b) Separation of singularities.** We introduce the partition of the unity:

$$1 \;\equiv\; \psi_0(x) \;+\; \psi_\infty(x) \;+ \sum_{k=1}^{m} \psi_k(x) \;,$$

where $\psi_k(x) \in C_0^\infty(R^1),\; k = 0, 1, \cdots, m;\; \psi_\infty(x) \in C^\infty(R^1)$ and $\psi_k(x) \equiv 1$ in a neighbourhood of both points $+x_k$ and $-x_k, k = 1, \cdots, m$, and $\psi_k(x) \equiv 0$ in a neighbourhood of both points $\pm x_j, j \neq k$, and at a neighbourhood of the points $x_0 = 0$ and $x_\infty = \infty$ . Similarly $\psi_0(x)$ and $\psi_\infty(x)$ are described.

We have

$$a(x) \;\equiv\; a_0(x) \;+\; a_\infty(x) \;+ \sum_{k=1}^{m} a_k(x) \;, \tag{3.4}$$

where $a_k(x) = a(x)\psi_k(x)\,(\,k = 0, 1, \cdots, m)$ and $a_\infty(x) = a(x)\psi_\infty(x)$ have a jump only at $\pm x_k$ and $x_\infty$, respectively. We denote by $K_{a_k}, k = 0, 1, \cdots, m$, the operators of the form (1.1) with $a_k(x)$ instead of $a(x)$ and similarly for $K_{a_\infty}$

**Lemma 3.2.** *The operator $K_a$ is Fredholm in the space $L_p(R^1, |x|^\gamma),\; 1 < p < \infty\;, -1 < \gamma < p-1$, iff the operators*

$$K_{a_0}\;,\; K_{a_1}\;, \cdots\;, K_{a_m}\;,\; K_{a_\infty}$$

*are Fredholm in this space. Besides this*

$$Ind\; K_a \;=\; Ind\; K_{a_\infty} \;+\; \sum_{j=0}^{m} Ind\; K_{a_j}\;. \tag{3.5}$$

Proof. The representation is valid

$$K_a = \lambda^{-n-2} K_{a_0} K_{a_\infty} \prod_{j=1}^{m} K_{a_j} + T , \tag{3.6}$$

where $T$ is a compact operator in $L_p(R^1, |x|^\gamma)$ and all the factors in the product commute up to a compact operator. To see the validity of (3.6), it suffices to consider the product of two operators $K_u$ and $K_v$ where $u(x)$ and $v(x)$ have jumps at different points. We have

$$K_u K_v = \lambda^2 K_{u+v} + P_+ u P_- \bar{v} P_+ . \tag{3.7}$$

Here and everywhere below we use the notation

$$\bar{v}(x) = v(-x).$$

By the Gohberg-Krupnik criterion on compactness of "composite" singular operators [5],[7]-[8], the operator $P_+ u P_- \bar{v} P_+$ is compact in $L_p(R^1, |x|^\gamma)$ . Therefore, from (3.7) we get (3.6) in view of (3.4). This gives the lemma's assertion. □

Thus, Lemma 3.2 shows that we are to study separately the following three "model" operators:

1. the operator $K_{a_0}$ with $a_0(x)$ having a jump only at the origin;
2. the operator $K_{a_\infty}$ with $a_\infty(x)$ having a jump only at infinity;
3. the operator $K_{a_k}$ with $a_k(x)$ having a jump only at the pair of the points $\pm x_k \neq 0$ , $k = 1, \cdots, m$.

   We find it convenient to single out another "model" case tightly connected with 1) and 2), although in view of Lemma 3.2 it reduces to these cases:

4. the operator $K_a$ with $a(x)$ having a jump only at the origin and infinity (of the type of *sign* $x$) with equal jumps : $a(+\infty) - a(-\infty) = a(+0) - a(-0)$, that is $s_o = s_\infty$.

**c) Symmetrizer.** We shall see below that the case 3) is covered by Theorem 2.16. As regards the cases 1) and 2) , an important role in their investigation is played by the direct connection between such cases realized by means of the operator which we call *symmetrizer*.

We denote

$$a^*(x) = - a\left(-\frac{1}{x}\right)$$

so that

$$a^{**}(x) \equiv a(x).$$

It is clear that if $a(x)$ has a jump only at infinity, then $a^*(x)$ has a jump only at the origin and *vice versa* .

**Definition 3.3.** The operator

$$K_{a^*} \;=\; \lambda I - P_+ a^*(x) P_- Q$$

will be called *symmetrizer* of the operator $K_a$.

The reason for the above definition becomes clear in view of the following lemma.

**Lemma 3.4.** *Let $a(x)$ have a jump only at the origin (or only at infinity). Then the relation holds*

$$K_a K_{a^*} \;=\; K_{a^*} K_a \;=\; \lambda^2 K_\omega \;+\; T \;,$$

*where $T$ is a compact operator in the space $L_p(R^1, |x|^\gamma)$,*

$$\omega(x) \;=\; a(x) \;+\; a^*(x) \;=\; a(x) \;-\; a\left(-\frac{1}{x}\right)$$

*and*

$$\omega(+\infty) - \omega(-\infty) \;=\; \omega(+0) - \omega(-0) \tag{3.8}$$

Proof. It suffices to refer to the general relation (3.7) and to the fact that $a(x)$ and $a^*(x)$ have jumps at different points which yields compactness of the last term in (3.7). □

**Lemma 3.5.** *Let $a(x) \in C(R^1)$ and have a jump only at infinity. The operators $K_a$ and $K_{a^*}$ are simultaneously Fredholm (invertible) in $L_2(R^1)$ and*

$$Ind\ K_a \;=\; Ind\ K_{a^*} \;.$$

Proof. Let

$$(R\varphi)(x) \;=\; \frac{1}{x}\varphi\left(\frac{1}{x}\right) \quad , \; R^2 = I. \tag{3.9}$$

The following relations are easily checked:

$$RS \;=\; -SR \quad , \quad RQ \;=\; -QR \;,$$

$$RP_+ \;=\; P_- R \quad , \quad RP_- \;=\; P_+ R \;.$$

Using these relations we verify the following connection between the operators $K_a$ and $K_{a^*}$ :

$$RQK_aQR \;=\; K_{a^*}. \tag{3.10}$$

The lemma's assertion follows immediately from (3.10). □

Thus, in the case $p = 2, \gamma = 0$ , the symmetrizer allows to reduce the consideration of the "model" cases 1) and 2) to the "symmetric" case 4). In the case $p \neq 2$ the connection (3.10) does not work since the operator $R$ is not bounded in the space $L_p(R^1, |x|^\gamma)$ except for the special case $\gamma = \frac{p}{2} - 1$ . By this reason for $p \neq 2$ we shall use another approach which is based on the exact calculation of the "main part" of the operator $K_a$ in the cases 1), 2) and 4).

### 3.3. Remarks (on invertibility).

We want to observe that the problem of precise describing of the exact spectrum instead of the essential spectrum of the operator $K_a$ is a hopeless affair in the sense that this problem includes as a particular case a problem of characterization of the spectrum of compact operators, although of a special form (with a kernel depending on a sum of arguments). To show this we give an example

$$K^\beta\varphi \ :\equiv\ \lambda\varphi(x) - \frac{c}{\pi}\int_0^\infty \frac{\varphi(y)\ dy}{x+y}\ +\ \beta\int_0^\infty e^{-(x+y)}\varphi(y)\ dy\ \ ,\ \ x>0\ ,c\geq 0. \tag{3.11}$$

By Theorem 2.6, the Carleman operator which is obtained from (3.11) in the case $\beta = 0$, is invertible (Fredholm) in $L_2(R^1_+)$ iff $\lambda \notin [0, c]$. It is easily shown that for any $\lambda_0 > c$ there exists a value of $\beta$ such that the operator $K^\beta$ is not invertible, having the deficiency numbers (1,1). Thus, the operator $K^\beta$ with this $\beta$ has the spectrum containing not only the continual part $[0, c]$, but also the point $\lambda_0 > c$ .

Now, it is clear that taking $e^{-(x+y)}P_m(x+y)$ instead of $e^{-(x+y)}$ in (3.11), $P_m$ being a polynomial, we may obtain in addition to the continual spectrum $[0, c]$ a finite number of isolated points of the spectrum.

## 4. Proof of Theorem A. The case $p = 2, \gamma = 0$.

### 4.1. The "model" operators 1) and 2).

The case 2) is covered by the following theorem.

**Theorem 4.1.** *Let $a(x) \in C(R^1)$ and $a(+\infty) \neq a(-\infty)$. The operator $K_a$ is Fredholm in the space $L_2(R^1)$ iff*

$$\lambda \notin [0, -\frac{s_\infty i}{2}], \tag{4.1}$$

*and then $Ind\ K_a =\ 0$.*

Proof. By Lemmas 3.4 and 3.5 the operator $K_a$ is simultaneously Fredholm with the operator $K_\omega$ and $Ind\ K_a = \frac{1}{2}\ Ind\ K_\omega$ . The function $\omega(x)$ has already equal jumps at the origin and infinity which allows to reduce the consideration of the operator $K_\omega$ to that of the Carleman operator (2.13). Indeed, we can put

$$\omega(x)\ =\ \Big(\omega(x)\ -\frac{s_\infty}{2}\ sign\ x\Big)\ +\ \frac{s_\infty}{2}\ sign\ x\ = \omega_1(x)\ +\ \omega_2(x),$$

so that the function $\omega_1(x)$ is continuous at $\dot{R}^1$. In the notation (1.1) we write

$$K_\omega\ =\ K_{\omega_2}\ -\ P_+\omega_1 P_- Q$$

where $P_+\omega_1 P_- Q$ is obviously compact. As regards the operator $K_{\omega_2}$, this is nothing else but the Carleman operator (2.13) with $\frac{1}{\pi}$ replaced by $\frac{s_\infty}{2\pi i}$ . Applying Theorem 2.6, we get (4.1) and the formula for the index. □

The case 1) immediately follows from Theorem 4.1 and Lemma 3.5 if we take into account that $a^{**}(x) = a(x)$ and $s_\infty(a^*) = s_0(a)$ . Instead of (4.1) we shall obtain

$$\lambda \notin \left[0, -\frac{is_0}{2}\right] .$$

### 4.2. The "model" operator 3).

This case is studied by means of Theorem 2.16.

**Theorem 4.2.** *Let $a(x) \in PC(R^1)$ and have a jump only at a pair of symmetric points $\pm x_k \neq 0$. Then the operator $K_a$ is Fredholm in the space $L_2(R_1)$ iff*

$$\lambda \notin \left[-\frac{is_k}{2}, \frac{is_k}{2}\right], \tag{4.2}$$

*and then Ind $K_a = 0$.*

To prove Theorem 4.2 we need the following lemma.

**Lemma 4.3.** *Let $a(x) \in PC(R^1)$ and have a jump only at a pair of symmetric points $\pm x_k \neq 0$. Then the operator $K_a$ is Fredholm in the space $L_p(R_1)$ , $1 < p < \infty$ iff the operator*

$$M = \lambda^2 I - P_+ a P_- \bar{a} P_+ \tag{4.3}$$

*is Fredholm in $L_p(R^1)$ and then*

$$Ind\ K_a = \frac{1}{2}\ Ind\ M . \tag{4.4}$$

Proof. We apply Theorem 2.16. In our case $A = \lambda^2 I$, $B = -P_+ a P_-$ and $Q\varphi = \varphi(-x)$ and the validity of Axioms 1-3 is well known. The operator $U$ required by Axiom 4 can be constructed as

$$U\varphi = u(x)\varphi(x) + i\, v(x) S\varphi ,$$

where

$$u(x) = \frac{x}{1+x^2} , \quad v(x) = \frac{|x^2 - x_j^2|}{1+x^2}.$$

It is Fredholm in $L_p(R^1)$ and satisfies the condition $UQ + QU = 0$. Consequently, Lemma 4.3 follows from Theorem 2.16. □

Proof. *(of Theorem 4.2.).* By Lemma 4.3 it suffices to study the operator $M$ . The latter is covered by Theorem 2.12. We observe that $\Delta(x) \equiv 1$ in our case and by (2.27) we have

$$h(x_k) = h(-x_k) = 2 + \frac{1}{\lambda^2}\left[a(x_k+0) - a(x_k-0)\right]\left[a(-x_k+0) - a(-x_k-0)\right]$$

$$= \ 2 \ + \ \left(\frac{s_k}{\lambda}\right)^2 \tag{4.5}$$

and

$$\nu_k^+ \ = \ \frac{h(x_k) \ + \ \sqrt{h^2(x_k) - 4}}{2} \ . \tag{4.6}$$

We apply Lemma 2.13 to the operator $M$ . The function $\omega(x)$ arising in this lemma has now the following form

$$\omega(x) \ = \ \left(\frac{x-i}{x+i}\right)_{x_k}^{-\frac{i}{2\pi i} \ ln \ \nu_k^+} \quad \left(\frac{x-i}{x+i}\right)_{-x_k}^{-\frac{i}{2\pi i} \ ln \ \nu_k^+} \tag{4.7}$$

in accordance with Remark 2.15. The operators $P_+\omega^{\pm 1}P_+ \ + \ P_-$ are Fredholm if $\arg \nu_k^+ \neq \pi(mod \ 2\pi)$ and even invertible if $-\pi < \arg \nu_k^+ < \pi$ . From (4.5)-(4.6) it follows that the last inequalities are nothing else but the condition (4.2). Under this condition, taking the invertibility of the operators $P_+\omega^{\pm 1}P_+ \ + \ P_-$ into account, we obtain $Ind \ M \ = \ 0$ and then $Ind \ K = 0$. □

## 5. Proof of Theorem A. The general case

### 5.1. The special case $a(x) = sign \ x$.

We find it convenient to consider this special case, which essentially illustrates the situation. The operator

$$K_a \ = \lambda I \ - \ P_+ \ sign \ x \ P_- Q \tag{5.1}$$

which formally corresponds to the Carleman operator (2.13), can be calculated implicitly as an integral operator with a homogeneous kernel of degree - 1 . Indeed, the following lemma is valid.

**Lemma 5.1.** *The operator (5.1) can be represented as*

$$K_a\varphi \ = \ \lambda\varphi(x) \ - \ \int_{-\infty}^{\infty} k(x,y)\varphi(y)dy \ , \ x \in R^1, \tag{5.2}$$

*where*

$$k(x,y) \ = \ \frac{\frac{1}{\pi i}(sign \ x \ + \ sign \ y) \ - \ \frac{2}{\pi^2} \ ln \left|\frac{y}{x}\right|}{4(x+y)} \ . \tag{5.3}$$

Proof. We have

$$K_a \ = \ \lambda I \ - \ \frac{1}{4}\left[(S \ sign \ x \ - sign \ x \ S) \ + \ (sign \ x \ I \ - \ S \ sign \ x \ S)\right] Q.$$

It remains to use the formula

$$(5.4) \qquad S\ sign\ x\ S\varphi \ = \ sign\ x\ \varphi(x) \ - \ \frac{2}{\pi^2}\int_{-\infty}^{\infty} \frac{ln|t| \ - \ ln|x|}{t-x}\varphi(t)\ dt$$

which is a corollary of the Poincaré-Bertrand formula. □

**Theorem 5.2.** *The operator* $K_a \ = \lambda I \ - \ P_+\ sign\ x\ P_- Q$ *is Fredholm (invertible ) in the space* $L_p\left(R^1, |x|^\gamma\right), \ -1 < \gamma < p-1$ *iff*

$$(5.5) \qquad \lambda \ \notin \ E_{\frac{1+\gamma}{p}}(-i) \ .$$

Proof. We apply Lemma 5.1 and Theorem 2.2. It is easy to check that the kernel (5.3) satisfies the summability conditions (2.7). Calculating the elements of the matrix (2.9), we have

$$(5.6) \qquad \begin{aligned} \mathcal{K}_{++}(z) &= \frac{1}{2\pi i}\ A(z) \ - \ \frac{1}{2\pi^2}\ A'(z)\ , \\ \mathcal{K}_{+-}(z) \ &= \ -\mathcal{K}_{-+}(z) \ = \frac{1}{2\pi^2}\ B(z)\ , \\ \mathcal{K}_{--}(z) &= \frac{1}{2\pi i}\ A(z) \ + \ \frac{1}{2\pi^2}\ A'(z)\ , \end{aligned}$$

where

$$A(z) \ = \ \int_0^\infty \frac{y^{z-1}\ dy}{y+1} \ = \ \frac{\pi}{\sin \pi z} \qquad \text{(see [9], 3.222.2)}\ ,$$

$$(5.7) \qquad A'(z) \ = \ \int_0^\infty \frac{y^{z-1}\ ln\ y\ dy}{y+1} \ = \ -\frac{\pi^2 \cos \pi z}{\sin^2 \pi z}\ ,$$

$$B(z) \ = \ \int_0^\infty \frac{y^{z-1}\ ln\ y\ dy}{y-1} \ = \ \frac{\pi^2}{\sin^2 \pi z}\ , \qquad \text{(see [9], 4.251.2)}\ .$$

Therefore, $\det \sigma(z) \ = \ \lambda\left(\lambda \ + \ \frac{i}{\sin \pi z}\right)$ and

$$(5.8) \qquad \det \sigma\left(ix + 1 - \frac{1+\gamma}{p}\right) \ = \ \lambda\left(\lambda + \frac{i}{\sin \pi(\frac{1+\gamma}{p} - ix)}\right)\ .$$

and then the condition (2.8) is nothing else but (5.5) in view of Lemma 2.5. Under this condition the operator $K_a$ is invertible by Theorem 2.2. □

## 5.2. The case of jump only at the origin

The idea of the direct calculation of the composition $P_+ a P_- Q$ as in the case $a(x) = sign\ x$ , may be applied in this case as well.

**Theorem 5.3.** *Let $a(x) \in C(\dot{R}^1 \backslash \{0\})$. The operator $K_a$ is representable in the form*

$$K_a\varphi \;=\; \lambda\varphi(x) \;-\chi(x)\int_{-1}^{1} k(x,y)\varphi(y)dy \;+\; T\varphi\;,\; x \in R^1, \tag{5.9}$$

*where $T$ is a compact operator in $L_p\left(R^1, |x|^\gamma\right)$, $\chi(x) = \chi_{[-1,1]}(x)$, and*

$$k(x,y) \;=\; s_0 \frac{\frac{1}{\pi i}(sign\ x \;+\; sign\ y) \;+\frac{2}{\pi^2}\ ln\left|\frac{y}{x}\right|}{8(x+y)}\;. \tag{5.10}$$

Proof. 1. Without loosing generality we may assume that $a(x)$ is an odd function such that

$$a(+0) \;=\; -\,a(-0)\;,\; a(\infty) = 0$$

(for this it suffices to observe that $K_a - K_b$ is compact if $a - b$ is continuous on $\dot{R}^1$).

2. Obviously, $K_a \;=\; \lambda I \;-\; \frac{1}{4}[(Sa - aS) \;+\; S(Sa - aS)]Q$. By the Poincaré-Bertrand formula we have

$$SaS\varphi \;=\; a(x)\;\varphi(x) \;-\; \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{\mathcal{A}(x) - \mathcal{A}(t)}{x-t}\varphi(t)\;dt\;,$$

where

$$\mathcal{A}(x) \;=\frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{a(t)}{t-x}dt. \tag{5.11}$$

By the assumptions on $a(x)$, the singular integral $\mathcal{A}(x)$ is well defined and is an even function with a logarithmic singularity at the origin :

$$\mathcal{A}(x) \sim \frac{2a(+0)}{\pi i}\;ln\frac{1}{|x|}\;,\quad x \to 0. \tag{5.12}$$

We denote for convenience

$$A\varphi \;=\; \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{a(x) - a(-y)}{x+y}\varphi(y)\;dy\;,$$

$$B\varphi = \;\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\mathcal{A}(x) - \mathcal{A}(-y)}{x+y}\varphi(y)\;dy\;,$$

so that

$$K_a \;=\; \lambda\varphi + A\varphi \;-\; B\varphi\;. \tag{5.13}$$

3. We shall show that the operators $A$ and $B$ admit the representation

$$A\varphi \;=\; \frac{a(+0)}{4\pi i}\chi(x)\int_{-1}^{1} \frac{sign\ x \;+\; sign\ y}{x \;+\; y}\varphi(y)\;dy \;+\; T_1\varphi \tag{5.14}$$

$$B\varphi \;=\; \frac{a(+0)}{2\pi^2}\chi(x)\int_{-1}^{1} \frac{ln\left|\frac{x}{y}\right|}{x \;+\; y}\;\varphi(y)\;dy \;+\; T_2\varphi\;, \tag{5.15}$$

where $T_1$ and $T_2$ are compact operators in $L_p\left(R^1,|x|^\gamma\right)$. Then from (5.13)-(5.15) the theorem's assertion will follow if we take into account that $s_0 = 2a(+0)$. To show (5.14)-(5.15) we observe that by the compactness of $bS - Sb$ in $L_p\left(R^1_+, x^\gamma\right)$ in case of a function $b(x)$ continuous on $[0,\infty]$ , the operator $A$ can be represented as

$$A\varphi = \frac{\theta_+(x)a(x)}{2\pi i}\int_0^\infty \frac{\varphi(y)}{x+y}\,dy + \frac{\theta_-(x)a(x)}{2\pi i}\int_0^\infty \frac{\varphi(-y)}{x-y}\,dy + T_3\varphi$$

where $\theta_\pm(x) = \frac{1}{2}(1 \pm sign\ x)$. Applying Theorem 2.4 with $a(\infty) = 0$ taken into account, we represent the operator $A$ as

$$A\varphi = \frac{\chi_+(x)a(+0)}{2\pi i}\int_0^1 \frac{\varphi(y)}{x+y}\,dy + \frac{\chi_-(x)a(-0)}{2\pi i}\int_0^1 \frac{\varphi(-y)}{x-y}\,dy + T_4\varphi$$

where $\chi_\pm(x) = \chi_{[0,\pm 1]}$ . This representation is nothing else but (5.14).

Passing to the operator $B$ , by (5.12) we have

$$\mathcal{A}(x) = \frac{2a(+0)}{\pi i}\, ln\frac{1}{|x|} + \beta(x) , \tag{5.16}$$

where $\beta(x) \in C(\dot{R}^1)$, $\beta(\infty) = 0$ and $\beta(x)$ is an even function. Since the operator $\beta S - S\beta$ is compact, the operator $B$ is reduced to

$$B\varphi = \frac{a(+0)}{2\pi^2}\int_{-\infty}^{\infty} \frac{\chi(x)\ ln\ |x| - \chi(y)\ ln\ |y|}{x+y}\varphi(y)\,dy + T_5\varphi ,$$

or

$$B\varphi = \frac{a(+0)}{2\pi^2}\chi(x)\int_{-\infty}^{\infty} \frac{ln\left|\frac{x}{y}\right|}{x+y}\varphi(y)\,dy + T_b\varphi + T_6\varphi , \tag{5.17}$$

where

$$T_b\varphi = \int_{-\infty}^{\infty} b(x,y)k(x,y)\varphi(y)\,dy,$$

with

$$b(x,y) = \frac{\chi(x)-\chi(y)}{ln\left|\frac{x}{y}\right|}\ ln\ |y| \ ,\ k(x,y) = \frac{ln\left|\frac{x}{y}\right|}{x+y} .$$

Here $k(x,y)$ is homogeneous and $b(x,y)$ is even in each variable, $|b(x,y)| \le 1$ everywhere and $b(x,y) \equiv 0$ if $|x| \le 1, |y| \le 1$ or $|x| \ge 1, |y| \ge 1$. Therefore, the operator $T_b$ is compact by Theorem 2.4. □

**Theorem 5.4.** *Let* $a(x) \in C(\dot{R}^1\backslash\{0\})$ . *The operator* $K_a$ *is Fredholm in* $L_p\left(R^1,|x|^\gamma\right)$ ,
$1 < p < \infty$, $-1 < \gamma < p-1$, *iff*

$$\lambda \notin E_{\frac{1+\gamma}{p}}\left(\frac{-is_0}{2}\right) \tag{5.18}$$

*and*

$$\text{(5.19)}\qquad Ind\ K_a \ = \begin{cases} 0, & \lambda \in \Omega_0^{ext} = \ exterior\ of\ E_{\frac{1+\gamma}{p}}\left(\frac{-is_0}{2}\right) \\ -\ sign\ (p-2-2\gamma)\ , & \lambda \in \Omega_0^{int} = \ interior\ of\ E_{\frac{1+\gamma}{p}}\left(\frac{-is_0}{2}\right) \end{cases}$$

*the set $\Omega_0^{int}$ being empty in the case $\gamma = \frac{p}{2} - 1$.*

Proof. In view of the representation (5.9)-(5.10), the operator $K_a$ can be treated by Theorem 2.3. We observe that the operator (5.9)-(5.10) has the same form as the operator (5.2), the difference being only in the sign inside the kernel. Therefore, to calculate $\sigma(z)$ in our case, we have only to change the sign in front of $A'(z)$ in (5.6), which gives

$$\text{(5.20)}\qquad \det \sigma \left( ix + 1 - \frac{1+\gamma}{p} \right) \ = \ \lambda \left( \lambda + \frac{is_0}{2 \sin \pi(\frac{1+\gamma}{p} - ix)} \right) \ .$$

Hence the condition (5.18) follows.

The formula for the index is obvious for $\lambda \in \Omega_0^{ext}$ because $\Omega_0^{ext}$ is connected and $K_a$ is invertible for large values of $|\lambda|$ . Let $\lambda \in \Omega_0$ . By Theorem 2.3 applied to (5.9) with (5.20) taken into account we have

$$\text{(5.21)}\qquad Ind\ K_a \ = \ -\ w\,(\lambda \ - f_\alpha(x)) \ , \quad \alpha = \frac{1+\gamma}{p} \ ,$$

where $w$ stands for the winding number and $f_\alpha(x) \ = \ \frac{is_0}{2\sin(\alpha - ix)\pi}$ . For $\lambda \in \Omega_0^{int}$ the curve

$$\text{(5.22)}\qquad z = \lambda - f_\alpha(x)\ , x \in R^1 \ ,$$

is the shifted lemniscate with 0 in its interior. Therefore, $Ind\ K_a \ = \ \pm\ 1$ . From (2.17) it is clear that the curve (5.22) is running in the positive (negative) direction if $B > 0 (B < 0$, resp.), that is, $\gamma < \frac{p}{2} - 1 (\gamma > \frac{p}{2} - 1$, resp.). Hence, by (5.21) $Ind\ K_a \ = \ \pm\ 1$ correspondingly to the cases $\gamma < \frac{p}{2} - 1$ and $\gamma > \frac{p}{2} - 1$ , which yields (5.19). □

### 5.3. The case of a jump only at infinity

We modify the relationship (3.10) so that it would work within the frameworks of weighted $L_p$-spaces. However, this modified connection will lead to the consideration of the operator $K_a$ and its symmetrizer $K_{a^*}$ in spaces with different weights.

We denote

$$R_\beta \varphi \ = \ \frac{sign\ x}{|x|^\beta} \varphi \left( \frac{1}{x} \right), \quad R_\beta^2 \ = \ I,$$

$$K_a^\nu \ = \ |x|^{1-\nu} K_a |x|^{\nu - 1} \ .$$

**Lemma 5.5.** *The operator $R_\beta$ is bounded in $L_p\left(R^1, |x|^\gamma\right)$, $-\infty < \gamma < \infty$, if*

$$\beta \ = \ \frac{2}{p}(1+\gamma)$$

*and* $K_a^{\nu}$, $-\infty < \nu < \infty$, *is bounded in* $L_p\left(R^1, |x|^\gamma\right)$ , *if*

$$p\nu \;-\; 1-p \;<\gamma \;<\; p\nu - 1\ . \tag{5.23}$$

Proof is direct.

**Corollary 5.6.** *Under the choice*

$$\nu = \frac{1+\beta}{2} = \frac{1}{2} \;+\; \frac{1+\gamma}{p} \qquad \left(\beta = \frac{2}{p}(1+\gamma)\right) \tag{5.24}$$

*the operators* $R_\beta$ *and* $K_a^{\frac{1+\beta}{2}}$ *are both bounded in* $L_p\left(R^1, |x|^\gamma\right)$ *for any* $\gamma \in (-\infty, \infty)$.

Indeed, in the case (5.24) the condition (5.23) is satisfied automatically.

**Lemma 5.7.** *The relationship*

$$R_\beta Q K_a^{\nu} Q R_\beta \;=\; K_{a^*}^{1+\beta-\nu} \tag{5.25}$$

*is valid.*

Proof. The formula of the "quasicommutation" is known:

$$R_\beta P_\pm \;=\; P_\mp^{\beta-1} R_\beta \tag{5.26}$$

where $P_\pm^{\beta-1} R_\beta = |x|^{1-\beta} P_\pm |x|^{1-\beta}$ , see [**?**] , formula (24.6). Using (5.26) and taking into account that

$$R_\beta\; aI \;=\; a\left(\frac{1}{x}\right) R_\beta\ , \quad QR_\beta \;=-\; R_\beta Q\ ,$$

we check (5.25) directly. □

**Corollary 5.8.** *The relation*

$$R_\beta Q K_a^{\frac{1+\beta}{2}} Q R_\beta \;=\; K_{a^*}^{\frac{1+\beta}{2}}$$

*holds where both* $R_\beta$ *and* $K_a^{\frac{1+\beta}{2}}$ *are bounded in the same space* $L_p\left(R^1, |x|^\gamma\right)$ , *if* $-\infty < \gamma < \infty$ *and* $\beta = \frac{2}{p}(1+\gamma)$.

**Theorem 5.9.** *Let* $a(x) \in C(R^1)$ *and have a jump only at infinity. The operator* $K_a$ *is Fredholm in* $L_p\left(R^1, |x|^\gamma\right)$, $1 < p < \infty$, $-1 < \gamma < p-1$, *iff*

$$\lambda \;\notin\; E_{\frac{1+\gamma}{p}}\left(-\frac{is_\infty}{2}\right) \tag{5.27}$$

*and*

$$Ind\ K_a \;=\begin{cases} 0, & \lambda \in \Omega_\infty^{ext} =\ exterior\ of\ E_{\frac{1+\gamma}{p}}\left(-\frac{is_\infty}{2}\right) \\ sign\ (p-2-2\gamma)\ , & \lambda \in \Omega_\infty^{int} =\ interior\ of\ E_{\frac{1+\gamma}{p}}\left(-\frac{is_\infty}{2}\right) \end{cases}$$

*the set* $\Omega_\infty$ *being empty in the case when* $\gamma = \frac{p}{2} - 1$.

Proof. From (5.25) we have

$$R_\beta Q K_a Q R_\beta \;=\; K_{a^*}^\beta\ , \quad \beta = \frac{2}{p}(1+\gamma). \tag{5.28}$$

Since Lemma 5.5 provides the boundedness of $R_\beta$ in $L_p\left(R^1, |x|^\gamma\right)$ , we observe from (5.28) that the operators $K_a$ and $K_{a^*}^\beta$ are simultaneously bounded (Fredholm, invertible) in $L_p\left(R^1, |x|^\gamma\right)$. The operator $K_a$ is bounded if $-1 < \gamma < p-1$. Therefore, $K_{a^*}^\beta$ is bounded in $L_p\left(R^1, |x|^\gamma\right)$ , or, which is the same, $K_{a^*}$ is bounded in $L_p\left(R^1, |x|^{\gamma^*}\right)$ , where

$$\gamma^* \;=\; p - 2 - \gamma\ .$$

Obviously, $\gamma \in (-1, p-1)$ is equivalent to $\gamma^* \in (-1, p-1)$ .

Since $a^*(x)$ has already a jump only at the origin with $a^*(+0) - a^*(-0) = s_\infty$, the Fredholmnes of $K_{a^*}$ in $L_p\left(R^1, |x|^{\gamma^*}\right)$ can be obtained from Theorem 5.4. According to (5.20)

$$\det \sigma \left( ix + 1 - \frac{1+\gamma^*}{p} \right) \;=\; \lambda \left( \lambda + \frac{is_\infty}{2 \sin \pi (ix + \frac{1+\gamma}{p})} \right)\ . \tag{5.29}$$

whence (5.27) follows. Observing that in (5.29) we have $-x$ instead of $x$ as we had in (5.20), we arrive at the formula for the index with the opposite sign in comparison with Theorem 5.4. □

## 5.4. The case of jumps only at a pair of points $\pm x_j \neq 0$ .

In this case Fredholmness of the operator $K_a$ does not depend on the weight exponent $\gamma$. Namely, the following lemma is valid.

**Lemma 5.10.** *Let* $a(x) \in PC(R^1)$ *have jumps only at a pair of symmetric points* $\pm x_j \neq 0$ . *The Fredholmness conditions and the index of the operator* $K_a$ *in* $L_p\left(R^1, |x|^\gamma\right)$, $-1 < \gamma < p-1$, *do not depend on* $\gamma$.

Proof.

1. We may choose $b(x) \in PC(R^1)$ with the same jumps that $a(x)$ has, so that $a(x) - b(x)$ is continuous on $\dot{R}^1$ and $b(0) = b(\infty) = 0$. Then $K_a - K_b$ is compact in $L_p\left(R^1, |x|^\gamma\right)$ .

2. Considering $K_b$ , we have for $\rho = |x|^{\frac{\gamma}{p}}$ :

$$\rho K_b \rho^{-1} \;-\; K_b \;=\; (\rho P_+ b \rho^{-1} - P_+ b)\rho P_- Q \rho^{-1} \;+\; P_+(b \rho P_- \rho^{-1} \;-\; b P_-)Q\ .$$

Here the differences in the parentheses are compact operators in $L_p(R^1)$ . Indeed,

$$(\rho P_+ b \rho^{-1} - P_+ b)\varphi \;=\; \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\left(\frac{|x|}{|y|}\right)^{\frac{\gamma}{p}} - 1}{y - x}\, b(y)\varphi(y)\ dy\ .$$

Since $b(0) = b(\infty) = 0$, the desired compactness follows from Theorem 2.4. Similarly, by means of Theorem 2.4, the second parentheses is treated.

Therefore, the operators $K_a, K_b$ and $\rho K_b \rho^{-1}$ are simultaneously Fredholm in $L_p(R^1)$, which proves the lemma. □

We shall treat the operator $K_a$ in $L_p(R^1)$ by means of Lemma 4.3 and the decomposition given in Lemma 2.13. Before that we need the following technical but important lemma on the connection between the values of $\lambda$ and $\arg \nu_k^+$, where $\nu_k^+$ are the numbers arising in the decomposition of Lemma 2.13. The domains $\Omega_k^{int}, k = 1, \cdots, m$, were defined in Subsection 3.1. Let $\Omega_k^{ext}$ be its exterior, that is the exterior of the two symmetrical lemniscates

$$E_{\frac{1}{p}}\left(-\frac{is_k}{2}\right) \bigcup E_{\frac{1}{p}}\left(\frac{is_k}{2}\right) .$$

**Lemma 5.11.** *Let $\nu_k^+ = \nu_k^+(\lambda)$ be the numbers (4.6) with $\arg \nu_k^+$ chosen according to (2.29). Then the condition*

$$(5.30) \qquad \left|\arg \nu_k^+\right| < \frac{2\pi}{\max(p,p')}$$

*is equivalent to the statement that*

$$\lambda \in \Omega_k^{ext}$$

*while the condition*

$$(5.31) \qquad \frac{2\pi}{\max(p,p')} < \left|\arg \nu_k^+\right| < \frac{2\pi}{\min(p,p')}$$

*is equivalent to $\lambda \in \Omega_k^{int}$ .*

Proof. From (4.6) we observe that $\nu_k^+$ is the root of the equation $\nu_k^+ + \frac{1}{\nu_k^+} = h(x_k)$. According to (4.5) this yields $\nu_k^+ + \frac{1}{\nu_k^+} - 2 = \left(\frac{s_k}{2}\right)^2$ or

$$\frac{s_k}{\lambda} = \pm\left(\sqrt{\nu_k^+} - \frac{1}{\sqrt{\nu_k^+}}\right) .$$

Putting $\nu_k^+ = re^{i\theta}$ , $\theta = \arg \nu_k^+$, we arrive at the relation

$$\lambda = \pm s_k \frac{te^{\frac{i\theta}{2}}}{t^2 e^{i\theta} - 1} , \quad t = \sqrt{r} > 0.$$

In the notation (2.21) this is

$$\lambda = \pm\frac{is_k}{2}\ell_{\frac{\theta}{2\pi}}(t) , \qquad \left(\ell_{-\alpha}(t) = -\ell_\alpha(\frac{1}{t})\right) .$$

In other words, given the fixed value of $\arg \nu_k^+ = \theta$, this is equivalent to saying that $\lambda$ belongs to

$$E_{\frac{\theta}{2\pi}}\left(-\frac{is_k}{2}\right) \bigcup E_{\frac{\theta}{2\pi}}\left(\frac{is_k}{2}\right) .$$

According to Corollary 2.10, the above union lies inside of the leaves of

$$E_{\frac{1}{p}}\left(-\frac{is_k}{2}\right) \bigcup E_{\frac{1}{p}}\left(\frac{is_k}{2}\right)$$

just under the condition (5.31). □

**Theorem 5.12.** *Let $a(x) \in PC(R^1)$ and have a jump only at a pair of symmetric points $\pm x_k \neq 0$. Then the operator $K_a$ is Fredholm in the space $L_p\left(R^1, |x|^\gamma\right)$ iff*

$$\lambda \notin E_{\frac{1}{p}}\left(\frac{-is_k}{2}\right) \bigcup E_{\frac{1}{p}}\left(\frac{is_k}{2}\right) \tag{5.32}$$

*and then*

$$Ind\ K_a = \begin{cases} 0\ , & \lambda \in \Omega_k^{ext} \\ sign\ (2-p)\ , & \lambda \in \Omega_k^{int} \end{cases} \tag{5.33}$$

Proof. In view of Lemma 5.10 we may take $\gamma = 0$ and use then Lemma 4.3. The operator $M$ arising in Lemma 4.3 is decomposed according to Lemma 2.13 (with Remark 2.15 taken into account) :

$$M = (P_+\frac{1}{\omega}P_+ + P_-)(P_+\omega P_+ + P_-) + T \tag{5.34}$$

because $\Delta(x) \equiv 1$ in our case. Here $\omega(x)$ has the form (4.7).

The operators $P_+\frac{1}{\omega}P_+ + P_-$ and $P_+\omega P_+ + P_-$ are Fredholm in $L_p(R^1)$ if $\arg \nu_k^+ \neq \frac{2\pi}{p}(mod\ 2\pi)$ and $\arg \nu_k^+ \neq \frac{2\pi}{p'}(mod\ 2\pi)$ . Hence the essential spectrum of the operator $M$ or, which is the same, of the operator $K_a$, is defined by the equality

$$\nu_k^+ = re^{i\theta}\ , \tag{5.35}$$

where $r > 0$ , $\theta = \frac{2\pi}{p}$ or $\theta = \frac{2\pi}{p'}$ . Similarly to the proof of Lemma 5.11 we have

$$\lambda = \pm s_k \frac{te^{\frac{i\theta}{2}}}{t^2e^{i\theta} - 1}\ , \quad \theta = \frac{2\pi}{p}\ , \tag{5.36}$$

where $t = \sqrt{r} > 0$. In the notation (2.21), this is

$$\lambda = \pm\frac{is_k}{2}\ \ell_{\frac{1}{p}}\ (t)\ , \quad t \in R^1_+.$$

in case of $\theta = \frac{2\pi}{p}$ . By Lemma 2.11 we get two loops of the lemniscate, that is $E_{\frac{1}{p}}\left(-\frac{is_k}{2}\right) \bigcup E_{\frac{1}{p}}\left(\frac{is_k}{2}\right)$ . As regards the second possibility $\theta = \frac{2\pi}{p'}$ , it gives the same loops, because these lemniscates are symmetric with respect to $p$ and $p'$ , see (2.18), where nothing changes if we replace $\alpha = \frac{1}{p}$ by $1 - \alpha$.

Hence, the essential spectrum of the operator $K_a$ is exactly $E_{\frac{1}{p}}\left(-\frac{is_k}{2}\right) \bigcup E_{\frac{1}{p}}\left(\frac{is_k}{2}\right)$ .

It remains to calculate the index of $K_a$ for $\lambda \notin E_{\frac{1}{p}}\left(-\frac{is_k}{2}\right) \bigcup E_{\frac{1}{p}}\left(\frac{is_k}{2}\right)$ . By Lemma 4.3 and the decomposition (5.34) we have

$$Ind\ K_a \ = \ \frac{1}{2}\ Ind\ M \ = \ -\frac{1}{2} ind_p\ \omega - \frac{1}{2}\ ind_p\ \frac{1}{\omega}.$$

To calculate the $p$-index of $\omega$ and $\frac{1}{\omega}$ , where $\omega$ is the function (4.7), we observe that the points $x \in R^1$ and $t = \frac{x-i}{x+i}$ run through the real axis $R^1$ and the circumference $|t| = 1$ , respectively, both in positive direction. Therefore,

$$ind_p \left(\frac{x-i}{x+i}\right)^{\beta} \ = \ ind_p t^{\beta}|_{along\ \ |t|=1} = \left[\Re\beta \ + \ \frac{1}{p'}\right] \ ,$$

where [ ] stands for the integer part of the number, see [?], pp. 17-18, Example 2.1. Consequently, for the function (4.7) we get

$$ind_p\ \omega = 2\left[\frac{1}{p'} - \frac{\arg \nu_k^+}{2\pi}\right] \ ,$$

$$ind_p\ \frac{1}{\omega} = 2\left[\frac{1}{p'} + \frac{\arg \nu_k^+}{2\pi}\right].$$

Under the choice (2.29) we obtain

$$ind_p\ \omega = 0\ , \quad ind_p\ \frac{1}{\omega} = \begin{cases} 0, & \text{if} \quad |\arg \nu_k^+| < \frac{2\pi}{\max(p,p')} \\ 2sign\ (p-2) & \text{otherwise} \end{cases}$$

after easy calculations. Therefore,

$$(5.37) \qquad Ind\ K_a \ = \ ind_p\ \frac{1}{\omega} = \begin{cases} 0, & \text{if} \quad |\arg \nu_k^+| < \frac{2\pi}{\max(p,p')} \\ -sign\ (p-2) & \text{otherwise} \end{cases}$$

It remains to refer to Lemma 5.11 to convert (5.37) into (5.36). □

Unifying Theorems 5.4, 5.9 and 5.12 on the basis of Lemma 3.2, we arrive at Theorem A .

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