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MULTI-DIMENSIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS

I. Introduction

This paper gives a self-contained representation of the Fredholm theory of one- and multi-dimensional integral equations

$$\lambda\varphi(x) - \int_{|y|<a} k(x,y)\varphi(y)\; dy\ ,\ x \in R^n,\ |x| < a\ ,\ 0 < a \leq \infty\ , \tag{1.1}$$

with the homogeneous kernel of degree $-n$, that is,

$$k(tx,ty) = t^{-n}k(x,y)\ ,\quad x,y \in R^n\ ,\quad t > 0.$$

Although known long ago, this theory was not exposed in its complete form anywhere, its initial results being presented in separate original papers L.Mikhailov [1]-[4]. In the one-dimensional case such equations (Mellin convolution equations) are well known as tightly connected with convolution type equations (Fourier convolution equations). In the multi-dimensional case the equations (1.1) may be effectively studied under the rotation invariance condition:

$$k\left(\omega(x),\omega(y)\right) = k(x,y)\ ,\quad x,y \in R^n\ , \tag{1.2}$$

where $\omega(x))$ is an arbitrary rotation in R^n,

Under this assumption the Fredholmness conditions in an explicit form can be given, as well as the formula for index. Using the apparatus of spherical harmonics, we reduce the equation (1.1) to a finite system of one-dimensional integral equations with a homogeneous kernel. This is the key moment of the investigation.

We present in a unified way both the known and new results, which is realized in terms of rigorous algebraic identities. This allows to consider also some kernels which do not satisfy the rotation invariance condition.

In Section 2 we treat separately the one-dimensional case. The main results on Fredholmness in the multi-dimensional case are presented in Section 3 , while Section 4 contains some further development for the multi-dimensional case (algebras, pseudospectra, projection method).

II. On operators with homogeneous kernels; the one-dimensional case

Integral operators

$$K\varphi \ :\equiv \ \int_0^a \ k(x,y)\varphi(y)\ dy \ , 0 < x < a \ , \ 0 < a \leq \infty \ , \tag{2.1}$$

with the kernel homogeneous of degree -1, that is,

$$k(tx,ty) = t^{-1}k(x,y)\ , \quad x,y \in R^1_+ \ , \quad t > 0, \tag{2.2}$$

may be considered as some counterparts of convolution operators. These operators in case of $a = \infty$, are invariant with respect to dilatations, not translations as convolutions are .

2.1. Connection with convolution operators; L_p-boundedness

A simple exponential change of variables establishes a direct correspondence between convolution type operators and operators (2.1). To show this, we introduce the notation for the following mapping which is an isometry between the L_p-spaces on $[0,a],\ 0 < a < \infty$, and R^1_+ (or on $[0,\infty]$ and R^1 in the case $a = \infty$) :

$$(W_p\varphi)(t) = e^{-\frac{t}{p}}\varphi(ae^{-t})\ , \quad 0 < t < \infty \ , \tag{2.3}$$

in the case $a < \infty$ and

$$(W_p\varphi)(t) = e^{-\frac{t}{p}}\varphi(e^{-t})\ , \quad -\infty < t < \infty \ , \tag{2.4}$$

in the case $a = \infty$. It is easy to see that

$$(W_p^{-1}\psi)(x) = \left(\frac{x}{a}\right)^{-\frac{1}{p}} \psi\left(-\ln\frac{x}{a}\right), \quad 0 < x < a, \tag{2.5}$$

in the case $0 < a < \infty$, and

$$(W_p^{-1}\psi)(x) = x^{-\frac{1}{p}}\psi\left(-\ln\ x\right), \quad 0 < x < \infty, \tag{2.6}$$

in the case $a = \infty$.

Lemma 2.1. *The operator W_p is an isometry of $L_p(0,a),\ 1 \leq p \leq \infty$, onto $L_p(R^1_+)$, when $0 < a < \infty$ and onto $L_p(R^1)$, when $a = \infty$, and*

$$W_pW_p^{-1} = I, \quad W_p^{-1}W_p = I\ . \tag{2.7}$$

Proof is obvious.

For a given kernel $k(x,y)$, homogeneous of degree -1, we define

$$h(t) = e^{\frac{t}{p'}}k(1,e^t)\ . \tag{2.8}$$

Let

$$H\varphi = \int_{-\infty}^{\infty} h(x-t)\varphi(t)dt \ , \quad x \in R^1 \ ,$$

and

$$\mathfrak{H}\varphi = \int_{0}^{\infty} h(x-t)\varphi(t)dt \ , \quad x \in R^1_+$$

be the convolution operators. The following relation provides a direct connection between the operator K and these operators:

$$W_p K W_p^{-1} = H, \ \text{ if } \ a = \infty \ , \quad \text{and} \ \ W_p K W_p^{-1} = \mathfrak{H}, \ \text{ if } \ a < \infty \ , \tag{2.9}$$

which can be verified directly.

We assume that

$$\kappa : = \int_{0}^{\infty} |k(1,y)| y^{-\frac{1}{p}} dy = \int_{0}^{\infty} |k(x,1)| x^{-\frac{1}{p'}} dx < \infty \ . \tag{2.10}$$

Theorem 2.2. *Let the kernel $k(x,y)$ be homogeneous of degree -1 and satisfy the condition (2.10). Then the operator K is bounded in $L_p(0,a), 1 \leq p \leq \infty, \ 0 < a \leq \infty$, with $\|K\| \leq \kappa$ and $\|K\| = \kappa$ in the case when $k(x,y) \geq 0$.*

Theorem 2.2 is known as Hardy-Littlewood theorem (see Hardy and Littlewood and Pólya [1]). We wish to remark that Theorem 2.2 is an immediate consequence of the relation (2.9).

Theorem 2.2 is easily extended to the case of weighted spaces

$$L_p\left([0,a], x^{\gamma}\right) \ = \left\{ f(x) : \ \int_{0}^{a} |f(x)|^p x^{\gamma} dx < \infty \right\} \ , \quad 0 < a \leq \infty \ , \tag{2.11}$$

with a power weight. In this case an isometry similar to that of Lemma 2.1 is valid, if we replace $\frac{1}{p}$ in (2.3) by $\frac{1+\gamma}{p}$. Since $\left(\frac{x}{y}\right)^{\frac{\gamma}{p}} k(x,y)$ is again a homogeneous kernel, we easily obtain the following result as a consequence of Theorem 2.2.

Theorem 2.3. *Let the kernel $k(x,y)$, homogeneous of degree -1, satisfy the condition*

$$\kappa_{\gamma} : = \int_{0}^{\infty} |k(1,y)| y^{-\frac{1+\gamma}{p}} dy \ < \infty \ . \tag{2.12}$$

Then the operator K is bounded in the space (2.11) with $\|K\| \leq \kappa_{\gamma}$ and $\|K\| = \kappa_{\gamma}$ in the case when $k(x,y) \geq 0$.

Remark 2.4. *In the case of the operator*

$$\int_{-a}^{b} k(x,y)\varphi(y)dy, \quad -a < x < b,$$

there holds a similar boundedness in the space $L_p\left([-a,b]; |x|^{\gamma}\right)$ with $0 < a \leq \infty, 0 < b \leq \infty$, if instead of (2.10) we require that

$$\int_{-\infty}^{\infty} |k(\pm 1,y)| y^{-\frac{1+\gamma}{p}} dy \ < \infty \ . \tag{2.13}$$

2.2. On Fredholmness of the operators $\lambda I - K$

a) The case of the equation on the half-axis. The equation

$$(\lambda I - K)\varphi \ :\equiv \ \lambda\varphi(x) - \int_0^\infty \ k(x,y)\varphi(y) \ dy \ , 0 < x < \infty \ , \tag{2.14}$$

by (2.9) is reduced to a convolution equation over the whole real line, which leads to the following theorem.

Theorem 2.5. *Let the kernel $k(x,y)$ satisfy the assumptions (2.2) and (2.10). Then the operator $\lambda I - K$ is Fredholm in $L_p\left(R^1_+, x^\gamma\right)$ if and only if it is invertible, and a necessary and sufficient condition for that is*

$$\sigma_K\left(ix + 1 - \frac{1+\gamma}{p}\right) \neq 0, \ x \in \dot{R}^1 \ , \tag{2.15}$$

where the symbol $\sigma_K(z)$ is defined by the Mellin transform of $k(1,y)$:

$$\sigma_K(z) \ = \ \lambda \ - \ \int_0^\infty k(1,y)y^{z-1}dy \ . \tag{2.16}$$

b) The case of a finite interval $[0,a]$. For a similar equation

$$(\lambda I - K)\varphi \ :\equiv \ \lambda\varphi(x) - \int_0^a \ k(x,y)\varphi(y) \ dy \ , 0 < x < a, \quad 0 < a < \infty \ , \tag{2.17}$$

on a finite interval, due to the connection (2.9) with a Wiener-Hopf operators we have the following theorem.

Theorem 2.6. *Let $k(x,y)$ satisfy the assumptions (2.2) and (2.10). Then the operator $\lambda I - K$ is Fredholm in $L_p([0,a], x^\gamma)$, if and only if the condition (2.15) is satisfied, where $\sigma_K(z)$ is the same as in (2.16), and then*

$$Ind \ (\lambda I - K) = -ind \ \sigma_K =: \varkappa$$

and $\alpha(\lambda I - K) = \max(0, \varkappa)$ and $\beta(\lambda I - K) = \max(0, -\varkappa)$.

c) The case of the whole line. For the equation

$$K\varphi \ :\equiv \ \lambda\varphi(x) - \int_{-\infty}^\infty \ k(x,y)\varphi(y) \ dy = \ f(x) \ , \ x \in R^1,$$

where

$$k(tx,ty) \ = \ t^{-1}k(x,y) \ , \quad x,y \in R^1 \ , \quad t > 0, \tag{2.18}$$

the following theorem is valid.

Theorem 2.7. *Let $k(x,y)$ satisfy the assumptions (2.18) and (2.13). Then the operator $\lambda I - K$ is Fredholm in the space $L_p\left(R^1, |x|^\gamma\right)$ if and only if it is invertible, and a necessary and sufficient condition for that is*

$$\det \sigma_K\left(ix + 1 - \frac{1+\gamma}{p}\right) \neq 0, \quad x \in \dot{R}^1 \ , \tag{2.19}$$

where

$$\sigma_K(z) \;=\; \begin{pmatrix} \lambda-\mathcal{K}_{++}(z) & -\,\mathcal{K}_{+-}(z) \\ -\,\mathcal{K}_{-+}(z) & \lambda-\mathcal{K}_{--}(z) \end{pmatrix} \tag{2.20}$$

and

$$\mathcal{K}_{\pm\pm}(z) = \int_0^\infty k(\pm 1, \pm y) y^{z-1}\, dy \ . \tag{2.21}$$

Proof. Theorem 2.7 is obtained by passing to the half-axes and using Theorem 2.5 relating to the half-axis. The justification of this passage is easily done by means of the relation

$$\begin{pmatrix} \theta_+ I & \theta_- I \\ \theta_- I & \theta_+ I \end{pmatrix}\begin{pmatrix} \lambda I - K & 0 \\ 0 & \lambda I \end{pmatrix}\begin{pmatrix} \theta_+ I & \theta_- I \\ \theta_- I & \theta_+ I \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda I - \theta_+ K\theta_+ & -\theta_+ K\theta_- \\ -\theta_- K\theta_+ & \lambda I - \theta_- K\theta_- \end{pmatrix} ,$$

where $\theta_\pm = \frac{1}{2}(1 \pm sign\ x)$ is the Heaviside function and the matrix operator

$$\begin{pmatrix} \theta_+ I & \theta_- I \\ \theta_- I & \theta_+ I \end{pmatrix}$$

is invertible: $Q^2 = I$. ■

d) The case of $[-a,b]$**.** For the equation

$$K\varphi :\ \equiv\ \lambda\varphi(x) - \int_{-a}^{b} k(x,y)\varphi(y)\ dy =\ f(x)\ ,\quad x\in[-a,b], \tag{2.22}$$

where $0<a<\infty, 0<b<\infty$, the following theorem is valid.

Theorem 2.8. *Let $k(x,y)$ satisfy the assumptions (2.18) and (2.13). Then the operator $\lambda I - K$ is Fredholm in $L_p([-a,b],|x|^\gamma)$, if and only if the condition (2.19) is satisfied, where $\sigma_K(z)$ is the same as in (2.20) and then*

$$Ind\ (\lambda I - K) = -ind\ \ (\det \sigma_K)\ \ .$$

We note some cases of Theorem 2.8, when we can give also an information about the deficiency numbers $\alpha(\lambda I - K)$ and $\beta(\lambda I - K)$. These are the cases when the kernel $k(x,y)$ is odd or even in one or both variables:

1) $k(x,y)$ *is even in* x $: k(x,y) = k(-x,y)$;
2) $k(x,y)$ *is even in* y $: k(x,y) = k(x,-y)$;
3) $k(x,y)$ *is odd in* x $: k(x,y) = -k(-x,y)$;
4) $k(x,y)$ *is odd in* y $: k(x,y) = -k(x,-y)$;
5) $k(x,y)$ *is even in* x *and* y *simultaneously* $: k(x,y) = k(-x,-y)$.

In the cases 1)-5) the matrix symbol (2.20) can be reduced to a triangular or even diagonal form.

We illustrate what happens in the case 1). In this case we have

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\sigma_K(z)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} =$$

$$= 2\begin{pmatrix} \lambda & 0 \\ -\mathcal{K}_{++}(z)+\mathcal{K}_{+-}(z) & \lambda-\mathcal{K}_{++}(z)-\mathcal{K}_{+-}(z) \end{pmatrix}.$$

Therefore, the necessary and sufficient conditions for Fredholmness of the operator $\lambda I - K$ are given by $\lambda\Big(\lambda-\mathcal{K}_{++}(z)-\mathcal{K}_{+-}(z)\Big)\neq 0$ for $z=i\xi+1-\frac{1+\gamma}{p}, \xi\in\dot{R}^1$. From the above matrix relation, we also derive immediately that

$$\alpha(\lambda I-K)=\max(0,\varkappa) \ \text{ and } \ \beta(\lambda I-K)=\max(0,-\varkappa)\ ,$$

where $\varkappa=-ind\,\left(\lambda-\mathcal{K}_{++}\left(i\xi+1-\frac{1+\gamma}{p}\right)-\mathcal{K}_{+-}\left(i\xi+1-\frac{1+\gamma}{p}\right)\right)$.

It should be noted that the matrix equality generates also an analogous operator equality.

In the case 5), for example, the matrix symbol (2.20) has a **circulant** form and we have the equality

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\sigma_K(z)\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}=$$

$$= 2\begin{pmatrix} \lambda-\mathcal{K}_{++}(z)+\mathcal{K}_{+-}(z) & 0 \\ 0 & \lambda-\mathcal{K}_{++}(z)-\mathcal{K}_{+-}(z) \end{pmatrix}.$$

with the diagonal matrix in the right-hand side. As a consequence from here we can easily calculate the deficiency numbers.

We note that the case

6) $k(x,y)$ is odd in x and y simultaneously $\ : k(x,y)=-k(-x,-y)$

is more difficult. In this case the reduction to a triangle matrix is also possible, but the resulting operator will include powers of operators over $[0,a]$, which disables us to derive an information about the deficiency numbers.

e) Compactness theorem. Let

$$T\varphi\ :\equiv\ \int_0^a\ c(x,y)k(x,y)\varphi(y)\ dy=\ f(x)\ ,x>0,\ \ 0<a\leq\infty, \tag{2.23}$$

where the homogeneous kernel $k(x,y)$ satisfies the assumptions (2.2) and (2.12).

Theorem 2.9. *Let $c(x,y)\in L_\infty([0,a]\times[0,a])$ and $c(+0,+0)=c(+\infty,+\infty)=0$ in the case $a=\infty$ and $c(+0,+0)=0$ in the case $0<a<\infty$, where the values $c(+0,+0)$ and $c(+\infty,+\infty)$ are understood in the sense*

$$c(+0,+0)=\lim_{N\to\infty}\ \underset{\substack{0<x<\frac{1}{N}\\ 0<y<\frac{1}{N}}}{esssup}\ |c(x,y)|\ ,\quad c(+\infty,+\infty)=\lim_{N\to\infty}\ \underset{\substack{x>N\\ y>N}}{esssup}\ |c(x,y)|\ , \tag{2.24}$$

then the operator T is compact in $L_p([0,a],|x|^\gamma)$.

Example 2.10. *The equation*

$$K_m\varphi:=\varphi(x)+b(x)\int_0^1\frac{c(y)\varphi(y)}{x^m+y^m}\ \varphi(y)dy\ ,\quad 0<x<1\ ,\quad m=1,2,... \tag{2.25}$$

is known as the Dixon equation in the case $b(x) \equiv c(y) \equiv$ *const and* $m = 1$ *(see Mikhailov [1] and the original paper Dixon [1]-[2]).*

To be able to apply Theorems 2.6 and 2.9 to the equation (2.25), we assume that the functions $x^{1-m}b(x)$ and $c(x)$ have a finite limit as $x \to 0$ in the sense similar to (2.24). We denote

$$\mu = \lim_{\substack{x\to 0\\ y\to 0}} \left[d(y)c(x)x^{1-m}\right] \ .$$

Considering Fredholmness of the operator (2.25), we may replace $b(x)c(y)$ by μx^{m-1} in view of Theorem 2.9, and thereby arrive at a kernel homogeneous of degree -1. By Theorem 2.6 the symbol of the obtained operator is

$$\sigma(z) = 1 + \mu \frac{\pi}{m} cosec \ \frac{\pi}{m} z \ .$$

By Theorem 2.6, the operator K_m is Fredholm in the space $L_p([0,1], x^\gamma), 1 < p < \infty, \ -p(m-1) - 1 < \gamma < p - 1$, if $\mu > -\frac{m}{\pi}$, when $\frac{1+\gamma}{p} = 1 - \frac{m}{2}$, and if $\mu \neq -\frac{m}{\pi} \sin\left[\frac{\pi}{m}\left(1 - \frac{1+\gamma}{p}\right)\right]$ when $\frac{1+\gamma}{p} \neq 1 - \frac{m}{2}$.

In the case $\frac{1+\gamma}{p} = 1 - \frac{m}{2}$, always $Ind\ \mathrm{K}_m = 0$.

In the case $\frac{1+\gamma}{p} \neq 1 - \frac{m}{2}$, to calculate the index we note that the range of the function $\sigma\left(i\xi + 1 - \frac{1+\gamma}{p}\right)$, when ξ runs R^1, is a closed loop in the right half-plane with the "end points" 0 and $cosec\left[\frac{\pi}{m}\left(1 - \frac{1+\gamma}{p}\right)\right] \in R^1_+$. This loop runs in the positive direction, if $\frac{m}{2} < 1 - \frac{1+\gamma}{p} < m$ and in the negative one, if $0 < 1 - \frac{1+\gamma}{p} < \frac{m}{2}$.

Therefore,

$$Ind\ \mathrm{K}_m = 0 \text{ if } \quad \mu > -\ \frac{m}{\pi} \sin\left[\frac{\pi}{m}\left(1 - \frac{1+\gamma}{p}\right)\right] \ .$$

In the case $\mu < -\ \frac{m}{\pi} \sin\left[\frac{\pi}{m}\left(1 - \frac{1+\gamma}{p}\right)\right]$ we obtain

$$ind\ \mathrm{K}_m = \begin{cases} 1\ , & \text{if} \quad 0 < 1 - \frac{1+\gamma}{p} < \frac{m}{2} \\ -1\ , & \text{if} \quad \frac{m}{2} < 1 - \frac{1+\gamma}{p} < m\ . \end{cases}$$

III. On operators with homogeneous kernels; the multi-dimensional case

The multi-dimensional version of the operator (2.1) has the form

$$K\varphi = \int_{B_a} k(x,y)\varphi(y)\ dy, \qquad x \in B_a, \tag{3.1}$$

where

$$B_a = \{y \in R^n : |y| \le a\}$$

is the ball in R^n of the radius a, where $0 < a \le \infty$.

In the case $n > 1$ we have no any simple connection between integral operators with homogeneous kernel and convolution operators, as we had in the one-dimensional case in the previous section. Such a connection may be, however, written for some other types of homegeneity, see Subsection 3.5 below.

We shall deal in the main with the following assumptions on the kernel $k(x, y)$:

1^0. *$k(x, y)$ is homogeneous of degree$-n$*, i.e.

$$k(tx, ty) = t^{-n}k(x, y), \quad \forall t > 0; \tag{3.2}$$

2^0. *$k(x, y)$ is invariant under the rotation group $SO(n)$*, i.e.

$$k(\omega(x), \omega(y)) = k(x, y), \quad \forall \omega \in SO(n). \tag{3.3}$$

But some of the results below will be given also in the case when we give up the rotation invariance condition (3.3).

Lemma 3.1. *Any function $k(x, y)$, defined on $R^n \times R^n$ and satisfying the rotation invariance condition 2^0 depends only on three scalar variables: $|x|, |y|$ and $x \cdot y$:*

$$k(x, y) = \ell(|x|^2, |y|^2, x \cdot y). \tag{3.4}$$

We refer the reader for the proof of Lemma 3.1 to Müller [1] or Samko [11], p.36, see also Weyl [1].

3.1. L_p-boundedness

We shall use the following *summation condition*

$$\kappa = \int_{R^n} |k(e_1, y)|\, |y|^{-\frac{n}{p}} dy < +\infty, \tag{3.5}$$

where

$$e_1 = (1, 0, \ldots, 0).$$

The following are examples of homogeneous kernels satisfying the above assumptions (3.2)-(3.3):

$$k_1(x, y) = \frac{1}{|x|^\alpha |x - y|^{n-\alpha}}, \qquad 0 < \alpha < n;$$

$$k_2(x, y) = \frac{1}{|x|^n + |y|^n} a\left(\frac{x \cdot y}{|x|\,|y|}\right) .$$

These examples satisfy the condition (3.5) for $p \in \left(1, \frac{n}{\alpha}\right)$ in the first case and for $p > 1$ in the second one for any $a(\sigma) \in L_1(S^{n-1})$.

Theorem 3.4 below gives the result on L_p-boundedness. To prove Theorem 3.3 we need the following auxiliary lemmas.

Lemma 3.2. *Under the assumptions (3.2)-(3.3) the integrals*

$$\kappa = \int_{R^n} |k(\sigma,y)|\,|y|^{-\frac{n}{p}}dy\ ,\ \ \sigma \in S^{n-1}, \tag{3.6}$$

and

$$\kappa_1 = \int_{R^n} |k(y,\theta)|\,|y|^{-\frac{n}{p'}}dy\ ,\quad \theta \in S^{n-1}\ , \tag{3.7}$$

where $0 < p \le \infty$*, do not depend on* $\sigma \in S^{n-1}$ *and* $\theta \in S^{n-1}$*, respectively.*

Proof. By

$$\omega_x(\eta)\ ,\quad \eta \in R^n\ , \tag{3.8}$$

we denote any rotation in R^n which transforms R^n onto itself so that

$$\omega_x(e_1) = \frac{x}{|x|}\ , \tag{3.9}$$

where x is a fixed vector in R^n. Evidently, such a rotation is unique in the case $n = 2$; in the case $n \ge 3$ there exist many such rotations and we choose any one of them. Obviously, for $\xi = \omega_x(\eta)$ we have

$$|\xi| = |\eta| \quad \text{and} \quad \xi \cdot \frac{x}{|x|} = \eta \cdot e_1 = \eta_1\ . \tag{3.10}$$

Making the rotation change of variables $y = \omega_\sigma(\tau)$ in (3.6), using the above properties of this rotation and the fact that $dy = d\tau$, we obtain the first statement of the lemma, since $k(\sigma, y)$ is rotation invariant. Similarly the change of variables $x = \omega_\theta(\tau)$, leads to the second statement. ■

Lemma 3.3. *Under the assumptions (3.2)-(3.3),* $\kappa = \kappa_1$ *.*

Proof. By Lemma 3.2 the integral (3.6) does not depend on $\sigma \in S^{n-1}$. Therefore,

$$\kappa = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} d\sigma \int_{S^{n-1}} d\theta \int_0^\infty |k(\sigma,\rho\theta)|\rho^{n-1-\frac{n}{p}}d\rho\ .$$

Making use of the homogeneity property of the kernel $k(x, y)$ and changing the variable $\rho = \frac{1}{r}$, we get

$$\kappa = \frac{1}{|S^{n-1}|}\int_{S^{n-1}} d\sigma \int_{S^{n-1}} d\theta \int_0^\infty |k(r\sigma,\theta)|r^{n-1-\frac{n}{p'}}dr\ .$$

Changing the order of integration in σ and θ by Fubini theorem, we see that the obtained inner integral is equal to the integral defined in (3.7). Taking Lemma 3.2 into account, we arrive at the equality $\kappa = \kappa_1$. ■

Theorem 3.4. *Let the kernel* $k(x, y)$ *satisfy the conditions* $1^0 - 2^0$ *and (3.5). Then the operator* K *is bounded in* $L_p(B_a), 1 \le p \le \infty,\ 0 < a \le \infty$ *and*

$$\|K\varphi\|_p \le \kappa\|\varphi\|_p. \tag{3.11}$$

If $k(x, y) \ge 0$*, the condition (3.5) is necessary for boundedness and* $\|K\|_{L_p\to L_p} = \kappa$.

Proof. We assume for simplicity that $a=\infty$ (if $a<\infty$, we may continue $\varphi(x)$ as zero beyond the ball B_a).

1). *Sufficiency part.* Applying the Hölder inequality, we obtain

$$|(K\varphi)(x)| \leq \left\{\int_{R^n} |y|^{-\frac{n}{p}}|k(x,y)|dy\right\}^{\frac{1}{p'}} \left\{\int_{R^n} |y|^{\frac{n}{p'}}|k(x,y)|\cdot|\varphi(y)|^p dy\right\}^{\frac{1}{p}} .$$

Making the change of variables $y \to |x|y$ in the first integral, taking the homogeneity property of the kernel $k(x,y)$ and Lemma 3.2 into account, we obtain

$$|(K\varphi)(x)| \leq \frac{\kappa^{\frac{1}{p'}}}{|x|^{\frac{n}{pp'}}} \left\{\int_{R^n} |y|^{\frac{n}{p'}}|k(x,y)|\cdot|\varphi(y)|^p dy\right\}^{\frac{1}{p}} .$$

Then

$$\|K\varphi\|_p \leq \kappa^{\frac{1}{p'}} \left\{\int_{R^n} |\varphi(y)|^p|y|^{\frac{n}{p'}}dy \int_{R^n} |k(x,y)|\cdot|x|^{-\frac{n}{p'}}dx\right\}^{\frac{1}{p}}$$

$$= \kappa^{\frac{1}{p'}} \left\{\int_{R^n} |\varphi(y)|^p dy \int_{R^n} \left|k\left(x,\frac{y}{|y|}\right)\right|\cdot|x|^{-\frac{n}{p'}}dx\right\}^{\frac{1}{p}} = \kappa^{\frac{1}{p'}}\kappa_1^{\frac{1}{p}}\|\varphi\|_p$$

due to the same Lemma 3.2. Taking Lemma 3.3 into account, we arrive at the estimate (3.11).

2). *Necessity part and calculation of the norm.* Let now the kernel be non-negative. Suppose that the operator K is bounded. Then

$$\left|\int_{R^n} (K\varphi)(x)\psi(x)dx\right| \leq \|K\|\cdot\|\varphi\|_p\|\psi\|_{p'} \tag{3.12}$$

for all $\varphi(x)\in L_p(R^n)$ and $\psi(x)\in L_{p'}(R^n)$. We choose

$$\varphi(x)=0, \ \text{ if } \ |x|<1, \quad \text{and} \quad \varphi(x)=|x|^{-\varepsilon-\frac{n}{p}}, \ \text{ if } \ |x|\geq 1,$$

and $\psi(x)=[\varphi(x)]^{p-1}$.

Substituting this into (3.10), we get

$$\int_{S^{n-1}} d\sigma \int_{R^n} k(\sigma,y)|y|^{-\varepsilon-\frac{n}{p}}dy \int\limits_{r>\max(1,|y|^{-1})} r^{-p\varepsilon-1}dr \leq \|K\|\cdot\|\varphi\|_p^p . \tag{3.13}$$

Direct calculation yields

$$\|\varphi\|_p^p = \frac{|S^{n-1}|}{p\varepsilon}, \qquad \int\limits_{r>\max(1,|y|^{-1})} r^{-p\varepsilon-1}dr = \frac{1}{p\varepsilon}\left[\max(1,|y|^{-1}\right]^{-p\varepsilon} ,$$

so that the inequality (3.13) takes the form

$$\frac{1}{|S^{n-1}|}\int_{S^{n-1}} d\sigma \int_{R^n} k(\sigma,y)|y|^{-\varepsilon-\frac{n}{p}}\left[\max(1,|y|^{-1}\right]^{-p\varepsilon}dy \leq \|K\| . \tag{3.14}$$

By Lemma 3.2 the inner integral in the left-hand side does not depend on σ, so that

$$\int_{R^n} k(e_1, y)|y|^{-\varepsilon-\frac{n}{p}} \left[\max(1, |y|^{-1}\right]^{-p\varepsilon} dy \le \|K\|. \tag{3.15}$$

Applying the Fatou theorem (see, e.g. Natanson [1] or Stein and Weiss [1]), we may pass to the limit as $\varepsilon \to 0$, which yields the inequality $\kappa \le \|K\|$. Together with the inverse inequality proved in the sufficiency part, this gives the equality $\|K\| = \kappa$. ■

The analysis of the proof of Theorem 3.4 shows that it may extended to the case when we have no the rotation invariance property (3.2). We may also consider the weighted case with the power weight $w(x) = |x|^\gamma, \gamma \in R^1$. Instead of (3.5), see also (3.6)-(3.7), we have to deal now with the constants

$$\kappa = \underset{\sigma\in S^{n-1}}{esssup}\ \kappa(\sigma), \quad \text{and} \quad \kappa_1 = \underset{\theta\in S^{n-1}}{esssup}\ \kappa_1(\theta) , \tag{3.16}$$

where

$$\kappa(\sigma) = \int_{R^n} |k(\sigma, y)|\, |y|^{-\frac{n+\gamma}{p}} dy \ , \ \sigma \in S^{n-1}, \tag{3.17}$$

and

$$\kappa_1(\theta) = \int_{R^n} |k(y, \theta)|\, |y|^{-\frac{n}{p'}+\frac{\gamma}{p}} dy \ , \quad \theta \in S^{n-1} \ . \tag{3.18}$$

The following result holds

Theorem 3.5. *Let the kernel $k(x, y)$ satisfy the homogeneity condition (3.2). If $\kappa < \infty$ and $\kappa_1 < \infty$, then the operator K is bounded in the space $L_p(B_a, |x|^\gamma),\ 1 \le p \le \infty$ and*

$$\|K\| \le \kappa^{\frac{1}{p'}} \kappa_1^{\frac{1}{p}} \ ,$$

where κ and κ_1 are defined in (3.16).

The interested reader can find the details of the proof of Theorem 3.5 in Karapetiants [1].

Remark 3.6. *Now, when the kernel $k(x, y)$ is not necessarily rotation invariant, the functions $\kappa(\sigma)$ and $\kappa(\sigma)$ may not coincide, as we had in Lemma 3.3, but their spherical means do coincide, that is,*

$$\int_{S^{n-1}} \kappa(\sigma) d\sigma = \int_{S^{n-1}} \kappa_1(\sigma) d\sigma,$$

see the proof of Lemma 3.3. In the case when $k(x, y)$ is non-negative, the conditions

$$\kappa(\sigma), \ \kappa_1(\sigma) \in L_1(S^{n-1})$$

are necessary for the operator K to be bounded in the space e $L_p(B_a, |x|^\gamma),\ 1 \le p \le \infty$

3.2. On spherical harmonics

We refer the reader to Müller [1], Stein [1], Stein and Weiss [1] for basics on spherical harmonics and remind here only the basic formulas we need to study the multi-dimensional equation (3.32), the Funk-Hekke formula being the most important for our goals.

Definition 3.7. *A polynomial $Y_m(x), x \in R^n$, of order m is called harmonic if $\Delta Y_m \equiv 0$, Δ being the Laplace operator. By H_m we denote the set of all homogeneous harmonic polynomials Y_m.*

A spherical harmonic $Y_m(\sigma), \sigma \in S^{n-1}$, of order m is defined as the restriction to the unit sphere S^{n-1} of a harmonic polynomial $Y_m \in H_m$.

Everywhere below we denote $x' = \frac{x}{|x|}$.

The dimension $d_n(m) = dim\ H_m$ of the space of spherical harmonics of order m is known to be equal to

$$d_n(m) = (n+2m-2)\frac{(n+m-3)!}{m!(n-2)!} \ , \tag{3.19}$$

so that $d_n(m) = 2$ in the case $n = 2$ and $d_n(m) \sim c\ m^{n-2}$ as $m \to \infty$, with $c = \frac{2}{(n-2)!}$.

Spherical harmonics of different orders are known to be orthogonal with respect to the scalar product

$$(u, v) = \int_{S^{n-1}} u(\sigma)v(\sigma)d\sigma \ .$$

The Fourier-Laplace decomposition of a function $f(\sigma)$ defined on S^{n-1} is given by

$$f(\sigma) \sim \sum_{m=0}^{\infty} Y_m(f, \sigma) \ , \tag{3.20}$$

where

$$Y_m(f, \sigma) = \sum_{\mu=1}^{d_n(m)} f_{m\mu} Y_{m\mu}(\sigma) \tag{3.21}$$

is the "m-th harmonic component" of the function $f(x)$. Here

$$f_{m\mu} = \int_{S^{n-1}} f(\sigma) Y_{m\mu}(\sigma) d\sigma \ ,$$

$Y_{m\mu}(\sigma)$ being any fixed orthonormal basis of spherical harmonics of order m. This component can be calculated by the formula

$$Y_m(f, x) = c_m \int_{S^{n-1}} f(\sigma) P_m(x \cdot \sigma) d\sigma, \qquad |x| = 1 \ , \tag{3.22}$$

where

$$c_m = \frac{d_n(m)}{|S^{n-1}|} \ , \tag{3.23}$$

and $P_m(t) = P_m(n, t)$ are the generalized Legendre polynomials normed by the condition $P_m(1) = 1$. We remind that

$$P_m(t) = \binom{m+n-3}{m}^{-1} C_m^{\frac{n-2}{2}} \ \text{ if } \ n \geq 3 \ \text{ and } \ P_m(t) = T_m(t) \ \text{ if } \ n = 2 \ ,$$

where $C_m^{\lambda}(t), \lambda = \frac{n-2}{2}$, and $T_m(t)$ are the Gegenbauer and Chebyshev polynomials, respectively. We remark that

$$|P_m(t)| \leq 1 \tag{3.24}$$

for all $m \in \mathbb{N}_0$ and $t \in [-1, 1]$.

The formula (3.25) below is known as the Funk-Hekke formula, see e.g. Müller [1].

Theorem 3.8. *Let* $a(t)(1-t^2)^{\frac{n-3}{2}} \in L_1([-1,1])$. *Then for any spherical harmonic* $Y_m(x)$, *the formula holds*

$$\int_{S^{n-1}} a(x\cdot\sigma)Y_m(\sigma)d\sigma = \lambda_m \; Y_m(x) \ , \quad |x| = 1 \ , \tag{3.25}$$

where

$$\lambda_m = |S^{n-2}| \int_{-1}^{1} a(t)P_m(t)(1-t^2)^{\frac{n-3}{2}} dt \ . \tag{3.26}$$

The integral operators on the sphere with a kernel depending on the inner product of arguments:

$$Af = \int_{S_{n-1}} a(x\cdot\sigma)f(\sigma)\; d\sigma \ , \quad x \in S^{n-1}. \tag{3.27}$$

usually are referred to as *spherical convolution operators* (Calderon and Zygmund [1]).

The operator (3.27) is well defined, for example, in the case

$$\int_{-1}^{1} |a(t)|(1-t^2)^{\frac{n-3}{2}} dt \ < \ \infty \ . \tag{3.28}$$

Since $\omega(x)\cdot\omega(\sigma) = x\cdot\sigma$, for any rotation, the operator (3.27) commutes with any rotation operator $Rf = f[\omega(\sigma)]$, that is, $KR = RK$.

In view of the Funk-Hekke formula (3.25), any spherical harmonic Y_m is an eigenfunction of the operator (3.27), say, under the assumption (3.28),

$$AY_m = \lambda_m Y_m \tag{3.29}$$

with λ_m given in (3.26).

Therefore, the operator A can be represented as

$$Af = \sum_{m=0}^{\infty} \lambda_m Y_m(f,x) \tag{3.30}$$

for nice functions $f(x)$, with $Y_m(f,x)$ defined in (3.21).

The spectrum $\{\lambda_m\}_{k=0}^{\infty}$ of the operator K is called its *spherical multiplier.*

Lemma 3.9. *Let (3.28) be satisfied. Then*

$$\lim_{m\to\infty} \lambda_m = 0. \tag{3.31}$$

3.3. Formal reduction to a system of one-dimensional equations with homogeneous kernels

We study the integral equation

$$\lambda\varphi(x) = \int_{B_a} k(x,y)\varphi(y)dy + f(x), \quad x \in B_a, \tag{3.32}$$

in the space $L_p(B_a),\ 1 \leq p \leq \infty$, where $k(x,y)$ satisfies the conditions (3.2)-(3.3), (3.5). The case $a = \infty$ is easier than the case of $a < \infty$, so we consider the case $a < \infty$. In view of Lemma 5.1, this equation has the form

$$\lambda\varphi(x) = \int_{B_a} \ell(|x|^2, |y|^2, x \cdot y)\varphi(y)dy + f(x), \quad x \in B_a, \tag{3.33}$$

where the kernel $\ell(r^2, \rho^2, r\rho t),\ r > 0, \rho > 0, t \in [-1,1]$, satisfies the condition

$$\mathfrak{L} := \int_0^\infty \int_{-1}^1 \left|\ell(1, \rho^2, \rho t)\right| \rho^{\frac{n}{p'}-1}(1-t^2)^{\frac{n-3}{2}} d\rho dt < \infty \quad , \mathfrak{L} = \frac{\kappa}{|S_{n-2}|}, \tag{3.34}$$

which is easily derived from (3.5) by passing to polar coordinates.

We wish to show that the equation (3.32) is reduced to an infinite system of one-dimensional equations (with a diagonal matrix) via the Funk-Hekke formula. We present some formalism of such a reduction in this subsection, and in the next subsection give a justification of this formalism.

We formally decompose the functions $\varphi(x) = \varphi(r\sigma)$ and $f(x) = f(r\sigma)$ into the Fourier-Laplace series (3.20) for a fixed value of r:

$$\varphi(x) = \varphi(r\sigma) = \sum_m^\infty \sum_{\mu=1}^{d_n(m)} \varphi_{m\mu}(r) Y_{m\mu}(\sigma)\ , \quad f(x) = f(r\sigma) = \sum_m^\infty \sum_{\mu=1}^{d_n(m)} f_{m\mu}(r) Y_{m\mu}(\sigma)\ , \tag{3.35}$$

where $\{Y_{m\mu}(\sigma)\}$ is any fixed orthonormal basis of spherical harmonics of order m and

$$\varphi_{m\mu} = \int_{S^{n-1}} \varphi(\sigma) Y_{m\mu}(\sigma) d\sigma \quad \text{and} \quad f_{m\mu} = \int_{S^{n-1}} f(\sigma) Y_{m\mu}(\sigma) d\sigma\ ,$$

are the Fourier-Laplace coefficients of the functions $\varphi(r\sigma)$ and $f(r\sigma)$ for a fixed value of $r > 0$.

We rewrite (3.33) in polar coordinates:

$$\lambda\varphi(r\sigma) = \int_0^a \int_{S_{n-1}} \ell(r^2, \rho^2, r\rho\sigma \cdot \theta)\rho^{n-1}\varphi(\rho\theta)\ d\rho\ d\theta + f(r\sigma). \tag{3.36}$$

Substituting the expansions (3.35) into (3.36) and applying the Funk-Hekke formula (3.25), after equalizing the coefficients in front of the spherical harmonics $Y_{m\mu}(\sigma)$, we arrive at the relations

$$\lambda\varphi_{m\mu}(r) - \int_0^a k_m(r,\rho)\varphi_{m\mu}(\rho)d\rho = f_{m\mu}(r), \tag{3.37}$$

$$m = 0, 1, 2, \cdots \quad \text{and} \quad \mu = 1, 2, \cdots, d_n(m)\ ,$$

where

$$k_m(r,\rho) = |S_{n-2}|\, \rho^{n-1} \int_{-1}^1 \ell(r^2, \rho^2, r\rho t) P_m(t)(1-t^2)^{\frac{n-3}{2}} dt =$$

$$= \frac{|S_{n-2}|}{\rho} \left(\frac{\rho}{r}\right)^n \int_{-1}^{1} \ell\left(1,\ \frac{\rho^2}{r^2},\ \frac{\rho}{r}t\right) P_m(t)(1-t^2)^{\frac{n-3}{2}} dt\ , \tag{3.38}$$

the latter relation being obtained due to the fact that the function $\ell(u, v, t)$ is a homogeneous function of its arguments: $\ell(\lambda u, \lambda v, \lambda t) = \lambda^{-\frac{n}{2}} \ell(u, v, t)$, which follows from the homegeneity of the kernel $k(x, y)$ and the equality $k(x, y) = \ell(|x|^2, |y|^2, x \cdot y)$.

Let us consider the one-dimensional integral operator

$$(K_m \psi)(r) = \int_0^a k_m(r, \rho)\psi(\rho) d\rho, \tag{3.39}$$

occurred in (3.37). Its kernel is homogeneous of order -1, as is seen from (3.38). Operators and equations with such operators were studied in the one-dimensional case in the preceding section.

It is clear via passage to polar coordinates, that if we study the initial multi-dimensional operator K in the space $L_p(B_a)$, the one-dimensional operators K_m must be considered in the weighted space $L_p([0, a], r^{n-1})$.

Lemma 3.10. *Let $k(x, y)$ satisfy the conditions (3.2)-(3.3), (3.5) and let K_m be the operators (3.39) with the kernel (3.38) related to the initial kernel $k(x, y)$ via the connection (3.4). Then*

$$\|K_m\| := \|K_m\|_{L_p([0,a],r^{n-1}) \to L_p([0,a],r^{n-1})} \to 0 \quad \textit{as} \ \to \infty, \quad 1 \le p \le \infty. \tag{3.40}$$

Proof. By Theorem 2.3 we have

$$\|K_m\| \le \int_0^\infty |k_m(1, \rho)| \rho^{-\frac{n}{p}} d\rho\ . \tag{3.41}$$

It is clear that

$$k_m(1, \rho) = |S_{n-2}|\, \rho^{n-1} \int_{-1}^{1} \ell\left(1,\ \rho^2,\ \rho t\right) P_m(t)(1-t^2)^{\frac{n-3}{2}} dt\ .$$

To prove (3.40), we apply the Lebesgue dominated convergence theorem. By (3.24) and (3.34), the function integrated in (3.41) has an integrable majorant not depending on m. Hence by the Lebesgue theorem it remains to show that $k_m(1, \rho) \to 0$ as $m \to \infty$ for almost all $\rho \in (0, \infty)$. Obviously, for any fixed ρ, the number $k_m(1, \rho)$, is nothing else but the multiplier (3.26) for the kernel

$$a(t) = |S_{n-2}|\, \rho^{n-1} \ell\left(1,\ \rho^2\ \rho t\right)\ .$$

Then this multiplier tends to zero by Lemma 3.9, the assumption (3.28) of that lemma being fulfilled due to (3.34) and Fubini's theorem. ■

Corollary. *There exists a number $M_0 \in \mathbb{N}$ such that the equation $(\lambda I - K_m)\psi = g,\ \lambda \neq 0$, with $g(r) \in L_p([0, a], r^{n-1}),\ 1 \le p \le \infty$, has the unique solution in $L_p([0, a], r^{n-1})$ for any $m > M_0$.*

3.4. Justification of the reduction and

the main result for rotation invariant homogeneous kernels

To the integral operator

$$(\lambda I-K)\varphi=\lambda\varphi(x)-\int_{B_a}k(x,y)\varphi(y)dy,$$

generated by the equation (5.32), we relate the following function sequence

$$\sigma_m(\xi)=\lambda-\int_{R^n}k(e_1,y)P_m(e_1\cdot y')|y|^{-n/p+i\xi}dy,\qquad m\in\mathbb{Z}_+\ ,\quad \xi\in\dot{R}^1\ , \tag{3.42}$$

which may be also represented in the form

$$\sigma_m(\xi)=\lambda-\int_0^\infty\int_{-1}^1\ell(1,\rho^2,\rho t)\rho^{\frac{n}{p'}-1+i\xi}P_m(t)(1-t^2)^{\frac{n-3}{2}}d\rho dt\ . \tag{3.43}$$

In the case when $\sigma_m(\xi)\neq 0$, $\forall\xi\in\dot{R}^1$ we denote

$$\varkappa_m=-\ ind\ \sigma_m(\xi)=-\frac{1}{2\pi}\Delta\Big[\arg\sigma_m(\xi)\Big]\Big|_{-\infty}^{\infty}.$$

We introduce also the following numbers

$$\alpha=\sum_{\varkappa_m>0}d_n(m)\varkappa_m\ ,\qquad \beta=-\sum_{\varkappa_m<0}d_n(m)\varkappa_m\ . \tag{3.44}$$

Remark 3.11. *The series in (3.44) are, in fact,finite sums under the assumptions (3.2)-(3.3),(3.5) since $\varkappa_m=0$ for large m by Lemma 3.10.*

Theorem 3.12. *The operator $\lambda I-K$ is Fredholm in the space $L_p(B_a)$, $0<a<\infty$, $1\le p\le\infty$, if and only if*

$$\sigma_m(\xi)\neq 0\quad m\in\mathbb{Z}_+,\quad \xi\in\dot{R}^1. \tag{3.45}$$

Under these conditions, the deficiency numbers $\alpha=\alpha(\lambda I-K)$ and $\beta=\beta(\lambda I-K)$ of the operators $\lambda I-K$ are computed by the formulas (3.44). In the case $B_a=R^n$, the word "Fredholm" may be replaced by the word "invertible" (with $\alpha=\beta=0$).

Proof. 1) *Passage to the matrix operator.* To justify the reduction to the system of one-dimensional equations (3.37), we introduce the following projectors.

$$P_{m\mu}\varphi=\varphi_{m\mu}(r)Y_{m\mu}(\sigma)\ ,\quad P_m\varphi=\sum_{\mu=1}^{d_n(m)}\varphi_{m\mu}(r)Y_{m\mu}(\sigma)\ , \tag{3.46}$$

and

$$\mathcal{P}_m\varphi=\sum_{j=0}^{m}\sum_{\mu=1}^{d_n(j)}\varphi_{j\mu}(r)Y_{j\mu}(\sigma)\ , \tag{3.47}$$

where $\varphi_{m\mu}$ are the Fourier-Laplace coefficients of the function $\varphi(r\sigma)$. They map the space $L_p(B_a)$ onto finite-dimensional subspaces in $L_p(B_a)$; in the case of the projector $\mathcal{P}_m$, for example, this subspace consists of functions of the form

$$\sum_{j=0}^{m} \sum_{\mu=1}^{d_n(j)} \varphi_{j\mu}(|x|) Y_{j\mu}\left(\frac{x}{|x|}\right) .$$

Evidently, $\varphi_{j\mu}(r) \in L_p\left([0,a], r^{n-1}\right)$. In the case $m = 0$, all the three projectors coincide and map $L_p(B_a)$ onto its subspace of radial functions.

Our justification will be based on the matrix identity

$$\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix} = \begin{pmatrix} P_+AP_+ + P_-BP_- & P_+AP_- + P_-BP_+ \\ P_-AP_+ + P_+BP_- & P_-AP_- + P_+BP_+ \end{pmatrix} , \tag{3.48}$$

where P_+, P_-, A and B are arbitrary operators. The particular case of such matrix identity was already used in the proof of Theorem 2.7. In the case when the operators P_+ and P_- in (3.48) are projectors, we have $P_+ + P_- = I$, and the matrix operator $\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}$ is invertible.

According to (3.48) we have

$$\begin{pmatrix} \mathcal{P} & I-\mathcal{P} \\ I-\mathcal{P} & \mathcal{P} \end{pmatrix} \begin{pmatrix} \lambda I - K & 0 \\ 0 & \lambda I \end{pmatrix} \begin{pmatrix} \mathcal{P} & I-\mathcal{P} \\ I-\mathcal{P} & \mathcal{P} \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda I - \mathcal{P}K\mathcal{P} & \mathcal{P}(\lambda I - K)(I-\mathcal{P}) \\ (I-\mathcal{P})(\lambda I - K)\mathcal{P} & \lambda I - (I-\mathcal{P})K(I-\mathcal{P}) \end{pmatrix} , \tag{3.49}$$

where $\mathcal{P}$ is any projector and K is any operator.

2) *Action of K on $P_m(L_p(B_a))$ and commutation relation.* We apply the matrix relation (3.49) to the projector $\mathcal{P} = \mathcal{P}_m$, defined in (3.47) and the operator K involved in (3.36), that is,

$$K\varphi = \int_0^a \int_{S_{n-1}} \ell(r^2, \rho^2, r\rho\sigma\cdot\theta)\rho^{n-1}\varphi(\rho\theta)\ d\rho\ d\theta\ , \quad x = r\sigma\ , \tag{3.50}$$

or, which is the same,

$$K\varphi = \int_0^a \int_{S_{n-1}} \frac{1}{\rho}\mathcal{L}\left(\frac{\rho}{r}, \sigma\cdot\theta\right)\varphi(\rho\theta)\ d\rho\ d\theta\ , \quad x = r\sigma\ , \tag{3.51}$$

where

$$\mathcal{L}(\rho, t) = \rho^n \ell(1, \rho^2, \rho t)\ . \tag{3.52}$$

We have

$$KP_{m\mu}\varphi = \mathfrak{K}_m P_{m\mu}\varphi \tag{3.53}$$

and

$$P_{m\mu}K\varphi = \mathfrak{K}_m P_{m\mu}\varphi, \tag{3.54}$$

where $\mathfrak{K}_m$ is the operator, the action of which on the subspace $P_{m\mu}$ is reduced to the application of the operator K_m, introduced in (3.39), to the radial factor, that is,

$$\mathfrak{K}_m\left(\varphi_{m\mu}Y_{m\mu}\right) = \left(K_m\varphi_{m\mu}\right)(r)Y_{m\mu}(\sigma)\ . \qquad (3.55)$$

Let us prove, for example, the formula (3.54):

$$P_{m\mu}K\varphi = (K\varphi)_{m\mu}Y_{m\mu}(\sigma) = Y_{m\mu}(\sigma)\int_0^a \rho^{n-1}d\rho\int_{S^{n-1}}\varphi(\rho\theta)d\theta\int_{S^{n-1}}\ell(r^2,\rho^2,r\rho\theta\cdot\xi)Y_{m\mu}(\xi)d\xi\ .$$

Applying the Funk-Hekke formula (3.25), we get

$$P_{m\mu}K\varphi = Y_{m\mu}(\sigma)\int_0^a k_m(r,\rho)\varphi_{m\mu}(\rho)d\rho = \mathfrak{K}_m P_{m\mu}\varphi\ .$$

Similarly the formula (3.53) can be proved.

Obviously, the commutation relations (3.53)-(3.54) hold for the projectors P_m as well. We note also that for $\mathcal{P}_m$ instead (3.53)-(3.54) we have

$$K\mathcal{P}_m\varphi = \sum_{j=1}^m \mathfrak{K}_j P_j\varphi \quad \text{and} \quad \mathcal{P}_m K\varphi = \sum_{j=1}^m \mathfrak{K}_j P_j\varphi$$

Therefore, we have

$$\mathcal{P}_m K(I-\mathcal{P}_m) = 0 \ \text{ and } \ (I-\mathcal{P}_m)K\mathcal{P}_m = 0\ . \qquad (3.56)$$

This means that the matrix identity (3.49) takes the form

$$\begin{pmatrix} \mathcal{P}_m & I-\mathcal{P}_m \\ I-\mathcal{P}_m & \mathcal{P}_m \end{pmatrix}\begin{pmatrix} \lambda I-K & 0 \\ 0 & \lambda I \end{pmatrix}\begin{pmatrix} \mathcal{P}_m & I-\mathcal{P}_m \\ I-\mathcal{P}_m & \mathcal{P}_m \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda I-\mathcal{P}_m K\mathcal{P}_m & 0 \\ 0 & \lambda I-(I-\mathcal{P}_m)K(I-\mathcal{P}_m) \end{pmatrix}\ . \qquad (3.57)$$

3) *Invertibility of the lower diagonal term.* We wish to show that the operator $\lambda I - (I-\mathcal{P}_m)K(I-\mathcal{P}_m)$ is invertible for large m. To this end, we intend to approximate the operator K by finite-dimensional ones under the operator norm. For the half-strip $\Pi = \{(\rho,t):\ 0<\rho<\infty,\ -1<t<1\}$, where the kernel $\mathcal{L}(\rho,t)$ of the operator K is defined, we consider the space

$$L_1(\Pi,w) = \left\{f(\rho,t):\ \int_\Pi |f(\rho,t)|\rho^{-\frac{n}{p}-1}(1-t^2)^{\frac{n-3}{2}}\ dt\ <\ \infty\right\},\quad w=\rho^{-\frac{n}{p}-1}(1-t^2)^{\frac{n-3}{2}}\ ,$$

generated by the condition (3.34). Let $C_0^\infty(\Pi)$ be the space of C^∞-functions with a compact support *supp* $f\subset\Pi$. It is well known that C_0^∞ is dense on $L_1(\Pi,w)$.

Therefore, we may approximate the kernel $\mathcal{L}(\rho,t)$ in the norm of the space $L_1(\Pi,w)$ by a C_0^∞-function $b(\rho,t)$ so that

$$\|\mathcal{L}-b\|_{L_1(\Pi,w)} < \varepsilon\ . \qquad (3.58)$$

Let

$$b(\rho,t)=\sum_{m=0}^{\infty} b_m(\rho)P_m(t) \ \in \ C_0^{\infty}(\Pi) \tag{3.59}$$

be the expansion of $b(\rho,t)$ into the series of Legendre polynomials $P_m(t)$, which converges in the norm of the space $L_1(\Pi,w)$, since the coefficients

$$b_m(\rho)=d_n(m)\frac{|S^{n-2}|}{|S^{n-1}|}\int_{-1}^{1} b(\rho,t)P_m(t)(1-t^2)^{\frac{n-3}{2}}\,dt\ , \tag{3.60}$$

having the compact support *supp* $b_m\subset[\delta,N]$, with $0<\delta<N<\infty$ not depending on m, admit the estimate

$$|b_m(\rho)|\le\frac{C}{m^q}\ , \tag{3.61}$$

for any $q=1,2,3,...$ with C not depending on m and ρ. The estimate (3.61) is easily derived from (3.60) via integration by parts due to the formula

$$P_m'(n,t)=\frac{m(m+n-2)}{n-1}P_{m-1}(n+2,t) \tag{3.62}$$

for the Legendre polynomials $P_m(t)=P_m(n,t)$.

Then we may choose M so that

$$\|b(\rho,t)-b_M(\rho,t)\|_{L_1(\Pi,w)}\ <\varepsilon\ , \tag{3.63}$$

where $b_m(\rho,t)=\sum_{m=0}^{M} b_m(\rho)P_m(t)$. Using the notation $K=K_{\mathcal{L}}$ for the operator K in dependence on its kernel, we consider the difference of operators

$$K-K_{b_M}=K_{\mathcal{L}}-K_{b_M}.$$

For the norm $\|K\|=\|K\|_{L_p(B_a)\to L_p(B_a)}$ we have

$$\|K\|\le\kappa=|S^{n-1}|\cdot\|\mathcal{L}\|_{L_1(\Pi,w)} \tag{3.64}$$

by (3.11) and (3.34). Therefore, from (3.58) and (3.63) we get

$$\|K-K_{b_M}\|_{L_p(B_a)\to L_p(B_a)}\le|S^{n-1}|\cdot\|\mathcal{L}-b_M\|_{L_1(\Pi,w)}<2|S^{n-1}|\varepsilon\ .$$

Then the operator $(I-\mathcal{P}_m)K(I-\mathcal{P}_m)$ involved in (3.57) may be represented as

$$(I-\mathcal{P}_m)K(I-\mathcal{P}_m)=(I-\mathcal{P}_m)(K-K_{b_M})(I-\mathcal{P}_m)+(I-\mathcal{P}_m)K_{b_M}(I-\mathcal{P}_m).$$

If we choose

$$m\ge M\ ,$$

then

$$(I-\mathcal{P}_m)K_{b_M}(I-\mathcal{P}_m)=0\ . \tag{3.65}$$

This is obvious, in fact, but may be also justified rigorously in the operator form, since

$$K_{b_M} = \sum_{j=0}^{M} \mathfrak{B}_j P_j,$$

where P_j are the projectors (3.46) and the operators $\mathfrak{B}_j\psi$ are similar to those appeared in (3.53)-(3.54) and are treated as in (3.55) with K_j replaced by $B_j\psi = c_j \int_0^a b_j\left(\frac{\rho}{r}\right)\frac{1}{\rho}\psi(\rho)d\rho$, with the constant c_j defined in (3.23). Then (3.65) becomes obvious due to the commutation relation (3.54).

In view of the formula (3.65), the matrix operator in the right-hand side of (3.57) is equal to

$$\mathbb{K} := \begin{pmatrix} \lambda I - \mathcal{P}_m K \mathcal{P}_m & 0 \\ 0 & \lambda I - (I-\mathcal{P}_m)(K-K_{b_M})(I-\mathcal{P}_m) \end{pmatrix}, \tag{3.66}$$

where $\|K - K_{b_M}\| < \varepsilon$. Choosing $\varepsilon < \frac{|\lambda|}{2|S^{n-1}|}$, we see that the operator $\lambda I - (I-\mathcal{P}_m)(K-K_{b_M})(I-\mathcal{P}_m)$ is invertible in the space $L_p(B_a)$, if $m \geq M$.

4) *The final part.* Because of the invertibility of the lower diagonal term in (3.57), we conclude from (3.57) and (3.66), that the operator K is Fredholm (or invertible) if the operator $\lambda I - \mathcal{P}_m K \mathcal{P}_m$ is Fredholm (invertible, respectively).

The Fredholmness properties of the operator $\lambda I - \mathcal{P}_m K \mathcal{P}_m$ are defined by those of the operator

$$\mathcal{P}_m(\lambda I - K)\Big|_{\mathcal{P}_m(L_p(B_a)} \tag{3.67}$$

The matrix operator (3.67) is reduced to a finite system of the equations (3.37) with $m = 0, 1, ..., M$. Applying the one-dimensional results given in Theorems 2.6 and 2.5 to each of the equations in (3.37), we arrive at the statement of the theorem. ■

3.5. Some cases of non-rotation invariant kernels and other types of homogeneity

a) The case of a kernel radial in x. Let now $k(x,y)$ in (3.32) be a radial function in x, but arbitrary in y, such that

$$k(x,y) = \ell(|x|, y) \ .$$

We introduce the symbol

$$\mathcal{A}(\xi) = \lambda - \int_{R^n} \ell(1,y)|y|^{-n/p+i\xi}dy, \quad \xi \in \dot{R}^1. \tag{3.68}$$

Let in (3.32) $a < \infty$ for definiteness.

Theorem 3.13. *Let the function $k(x,y) = \ell(|x|, y)$ satisfy the homogeneity condition (3.2) and the condition*

$$\underset{\theta\in S^{n-1}}{esssup} \ \ \kappa_1(\theta) < \infty \ , \tag{3.69}$$

where $k_1(\theta)$ has the form (3.18) with $\gamma = 0$. Then the operator $\lambda I - K$ is Fredholm in $L_p(B_a), 1 \leq p \leq \infty$, if and only if $\sigma(\xi) \neq 0, \quad \xi \in \dot{R}^1$, and $Ind\ (\lambda I - K) = -ind\ \mathcal{A}(\xi)$.

Proof. First of all we note that the operator K is bounded in the space $L_p(B_a)$ under the assumptions of the theorem. Indeed, the boundedness conditions are given by Theorem 3.5. The function $\kappa(\sigma)$ introduced in (3.17) is constant in this case and similarly to the proof of Lemma 3.3, it can be shown that the condition $\kappa(\sigma) = \kappa < \infty$ follows automatically from the condition (3.69).

Let $P_0 = \mathcal{P}_0$ be the projector defined in (3.46) mapping $L_p(B_a)$ onto its subspace of radial functions. Evidently,

$$(I - \mathcal{P}_0)K = 0 \ , \qquad (3.70)$$

because of which the matrix identity (3.49) in this case takes a simpler form

$$\begin{pmatrix} \mathcal{P}_0 & I-\mathcal{P}_0 \\ I-\mathcal{P}_0 & \mathcal{P}_0 \end{pmatrix} \begin{pmatrix} \lambda I - K & 0 \\ 0 & \lambda I \end{pmatrix} \begin{pmatrix} \mathcal{P}_0 & I-\mathcal{P}_0 \\ I-\mathcal{P}_0 & \mathcal{P}_0 \end{pmatrix} =$$

$$= \begin{pmatrix} \lambda I - \mathcal{P}_0 K \mathcal{P}_0 & -\mathcal{P}_0 K (I-\mathcal{P}_0) \\ 0 & \lambda I \end{pmatrix} .$$

This matrix identity immediately yields the statement of the theorem in view of the one-dimensional result presented in Theorem 2.6. ■

The case of the kernel radial in y may be treated similarly.

b) The case of coordinate-wise homogeneity.

Saying that $k(x, y)$ is a coordinate-wise homogeneous kernel of vector degree $-\mathbf{1} = (-1, -1, ..., -1)$, we mean that

$$k(\lambda \circ x, \lambda \circ y) = \lambda^{-1} k(x, y) \ ,$$

where $\lambda \in R^n, \ \lambda > 0$ and we denote

$$\lambda \circ x = (\lambda_1 x_1, \lambda_2 x_2, ..., \lambda_n x_n) \ ,$$

$$\lambda^{-\mathbf{1}} = \lambda_1^{-1} \cdot \lambda_2^{-1} ... \cdot \lambda_n^{-1} .$$

Let $j_u, \ u = 1, 2, ..., 2^n$, be the vertices of the cube $|x| \le 1$ and Γ_u the corresponding octant in R^n containing a vertex j_u.

Theorem 3.14. *Let $k(x, y)$ be coordinate-wise homogeneous kernel and let*

$$k_{u,v} = \int\limits_{\Gamma_v} |k(j_u, \tau)| \prod_{j=1}^{n} |\tau_j|^{-\frac{1}{p}} d\tau < \infty, \quad u, v = 1, 2, \dots, 2^n. \qquad (3.71)$$

Then the operator K is bounded in $L_p(R^n), 1 \le p \le \infty$.

We refer to Karapetiants [2] for the proof of Theorem 3.14.

Integral equations with coordinate-wise homogeneous kernels can be treated by means of the Lemma below on operators with projectors.

Let $P_1, P_2, ..., P_n$ be mutually orthogonal projectors in the Banach space X:

$$P_1 + P_2 + \cdots + P_n = I; \quad P_j^2 = P_j; \quad P_i P_j = 0, \ i \ne j, \ i, j = 1, 2, ..., n \ ,$$

A_j be bounded linear operators in X, $j = 1, 2, ..., n$ and

$$\Pi = \sum_{j=1}^{n} P_j A_j \ .$$

Lemma 3.15. *The operator* Π *is Fredholm (invertible) in* X *if and only if the matrix operator*

$$\mathbb{W} = \Big(P_j A_j P_r \ + \ \delta_{rj}(I - P_j) \Big)_{j,r=1}^{n} \tag{3.72}$$

is Fredholm (invertible, resp.) in $X^n = X \times X \times \cdots \times X$ *, where* $\delta_{rj} = 1$*, if* $r = j$ *and* $\delta_{rj} = 0$ *otherwise.*

Proof. Let

$$\mathbb{V} = (P_{r+j-1})_{j,r=1}^{n}, \ P_{n+j} = P_j \ \text{ and } \ \mathbb{U} = \begin{pmatrix} \Pi & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix} ,$$

where $\mathbb{I}_{n-1}$ is the identity operator in X^{n-1}. We have $\mathbb{W} = \mathbb{VUV}$. Hence the lemma's statement follows, since $\mathbb{V}^2 = \mathbb{I}$. ■

Applying this Lemma, for example, to the equation

$$\lambda\varphi(x) - \int\limits_{R^n} k(x,y)\varphi(y)dy = f(x), x \in R^n, \tag{3.73}$$

we obtain the following theorem.

Theorem 3.16. *Let* $k(x,y)$ *be coordinate-wise homogeneous kernel of vector degree* $-\mathbf{1}$ *and let the conditions (3.70) be satisfied. For the operator* $\lambda I - K$ *to be invertible in the space* $L_p(R^n)$, $1 \le p \le \infty$, *it is necessary and sufficient that*

$$\det \left\{ \lambda E - \left(\mathcal{M}_{u,v}(i\xi + \frac{1}{p'} j_1) \right)_{u,v=1}^{2^n} \right\} \neq 0, \quad \xi \in R^n, \tag{3.74}$$

where $\mathcal{M}(z)$ *is the Mellin transform*

$$\mathcal{M}_{u,v}(z) = \int\limits_{\Gamma_1} k(j_u, j_v \circ y) y^{z-\mathbf{1}} dy, \quad u,v = 1,2,\ldots,2^n.$$

and $\Gamma_1 = \{x \in R^n : x_1 > 0, ..., x_n > 0\}$, $j_1 = \mathbf{1}, y^{z-\mathbf{1}} = y_1^{z_1 - 1} \cdots y_n^{z_n - 1}$.

IV. Some further developments

4.1. The algebra of multi-dimensional integral operators

According to Theorem 3.12 we can relate, with any operator $\lambda I - K$, where K is of the form (3.32), the function

$$\sigma(m,\xi) = \lambda - \int\limits_{R^n} k(e_1, y) P_m(e_1 \cdot y)|y|^{-\frac{n}{p}+i\xi} dy, \tag{4.1}$$

defined on the set $\mathbb{Z}_+\dot{\times}R$. The function (4.1) is called the symbol of the operator $\lambda I - K$.

The following is the reformulation of Theorem 3.12.

Theorem 4.1. *The operator $\lambda I - K$ is Fredholm in $L_p(B_n)$ if and only if the function $\sigma(m,\xi)$ is nonvanishing, i.e.*

$$\sigma(m,\xi) \neq 0, \qquad \forall (m,\xi) \in \mathbb{Z}_+\dot{\times}R. \tag{4.2}$$

If the condition (4.2) is satisfied, then the index of $\lambda I - K$ can be computed by the formula:

$$Ind\ \ (\lambda I - K) = -\sum_{m=0}^{\infty} d_n(m)\ ind_\xi\ \sigma(m,\xi), \tag{4.3}$$

where $ind_\xi \sigma(m,\xi)$ is the winding number of the function $\sigma(m,\xi)$ with fixed m.

Let $\mathcal{K}$ be the smallest closed subalgebra of the Banach algebra $\mathcal{L}(L_p)$ containing all operators of the form $\lambda I - K$, with $\lambda \in \mathbb{C}$. It represents the closure in the operator norm of the set

$$\mathcal{K}_0 = \left\{ \sum_i \prod_j (\lambda_{ij} I - K_{ij}) \right\},$$

where the sum and the products are finite. Let $\mathcal{T}$ be the set of compact operators containing in the algebra $\mathcal{K}$. Clearly, $\mathcal{T}$ is a closed two-sided ideal of $\mathcal{K}$. So we may consider the quotient algebra $\mathcal{K}\backslash\mathcal{T}$. It is obvious, that every element of $\mathcal{K}\backslash\mathcal{T}$, containing at least one element of $\mathcal{K}_0$, has actually the form $\lambda I - K + \mathcal{T}$ and the set

$$(\mathcal{K}\backslash\mathcal{T})_0 = \{\lambda I - K + \mathcal{T}\}$$

is dense in the algebra $\mathcal{K}\backslash\mathcal{T}$.

It is easy to see that the algebra $\mathcal{K}\backslash\mathcal{T}$ is commutative. Indeed, using the known results on compactness, one can show that if K_1 and K_2 are of the form (3.6), then $[K_1, K_2] = K_1K_2 - K_2K_1 \in \mathcal{T}$.

Now let us study the invertibility in the algebra $\mathcal{K}\backslash\mathcal{T}$. With every element $\lambda I - K + \mathcal{T} \in (\mathcal{K}\backslash\mathcal{T})_0$ we connect the function, defined by (4.1). We call this function the symbol of the element $\lambda I - K + \mathcal{T}$. It is easy to prove, that the sum and the product of symbols correspond to the sum and the product of elements of $(\mathcal{K}\backslash\mathcal{K})_0$, respectively. The following proposition will be used below.

Lemma 4.2. *For every $\lambda I - K + \mathcal{T} \in (\mathcal{K}\backslash\mathcal{T})_0$ and its symbol $\sigma(m,\xi)$ the inequality*

$$\sup_{\mathbb{Z}_+\dot{\times}R^1} |\sigma(m,\xi)| \leq \|\lambda I - K + \mathcal{T}\|_{\mathcal{K}\backslash\mathcal{T}}$$

holds.

Proof can be obtained from Theorem 4.1 by standart arguments.

Let $C\left(\mathbb{Z}_+\dot{\times}R^1\right)$ be the Banach space of all continuous functions defined on the compact $\mathbb{Z}_+\dot{\times}R^1$ with the norm

$$\|f(m,\xi)\|_C = \sup_{\mathbb{Z}_+\dot{\times}R^1} |f(m,\xi)|\,.$$

Let us associate with every element of $\mathcal{K}\backslash\mathcal{T}$ a certain function of $C\left(\mathbb{Z}_+\dot{\times}R^1\right)$.

Theorem 4.3 *Let $A+\mathcal{T}\in\mathcal{K}\backslash\mathcal{T}$. The element $A+\mathcal{T}$ is invertible in the algebra $\mathcal{K}\backslash\mathcal{T}$ if and only if its symbol is non-degenerate, i.e.*

$$\sigma(m,\xi)\neq 0, \quad \forall(m,\xi)\in\mathbb{Z}_+\dot{\times}R^1.$$

Definition 4.4. *The symbol of the operator $A\in\mathcal{K}$ is a function $\sigma_A(m,\xi)$ which is the symbol of the coset $\{A+\mathcal{T}\}\in\mathcal{K}\backslash\mathcal{T}$.*

From Theorem 4.3 we easy derive the following result.

Theorem 4.5. *Let $A\in\mathcal{K}$. The operator A is a Fredholm operator if and only if*

$$\sigma_A(m,\xi)\neq 0, \quad \forall(m,\xi)\in\mathbb{Z}_+\dot{\times}R^1. \tag{4.4}$$

If (4.4) holds, then the index of A can be computed by the formula

$$Ind\ A = -\sum_{m=0}^{\infty} d_n(m)\ ind_\xi\ \sigma_A(m,\xi). \tag{4.5}$$

4.2. Finite section method

Let X be a Banach space and $A\in\mathcal{L}(X)$. Suppose $\{P_\tau\}_{\tau\in(0,1)}\subset\mathcal{L}(X)$ and $\{Q_\tau\}_{\tau\in(0,1)}\subset\mathcal{L}(X)$ are families of projections with the property that $P_\tau\to I$ and $Q_\tau\to I$ in the strong sense as $\tau\to 0$. Now consider the equation

$$Q_\tau A P_\tau x = Q_\tau y. \tag{4.6}$$

Definition 4.6. *They say that the finite section method is applicable to an operator A if*
1) there exists a number $\tau_0\in(0,1)$ such that for each $y\in X$ the equation (4.6) has the unique solution $x\in P_\tau X$ for all $\tau<\tau_0$;
2) x_τ converges in the norm of X to a solution $x\in X$ of the equation $Ax=y$ as $\tau\to 0$.

If the finite section method is applicable to A then we write $A\in\Pi\{P_\tau\}$. For details about the finite section method see Gohberg and Fel'dman [1], Böttcher and Silbermann [1] and Prössdorf and Silbermann [1].

Now let us consider the projection on $L_p(B_n)$, $1\leq p<\infty$, defined by

$$\left(P_\tau\varphi\right)(x) = \begin{cases} \varphi(x) & , \quad \tau<|x|\leq 1 \\ 0 & , \quad |x|\leq\tau \end{cases}.$$

We are interested in answering the question: whether the finite section method is applicable to the operator $\lambda I - K$. The following proposition provides the desired answer.

Theorem 4.7. *Let the operator $\lambda I - K$ be bounded in $L_p(B_n)$, $1 \le p < \infty$. Then $\lambda I - K \in \Pi\{P_\tau\}$ if and only if $\lambda I - K$ is invertible.*

The operator $K_\tau = P_\tau K P_\tau$ is called the truncated operator. Let R_τ be the operator defined by the formula

$$(R_\tau\varphi)(x) = \begin{cases} \left(\dfrac{\tau}{|x|^2}\right)^{n/p} \overline{\varphi\left(\tau \dfrac{x}{|x|^2}\right)} & , \quad \tau < |x| \le 1 \\ 0 & , \quad |x| \le \tau \end{cases} .$$

It is easy to prove the following properties of R_τ :
1) for every $\tau \in (0,1)$ we have $R_\tau^2 = P_\tau$ and $R_\tau P_\tau = P_\tau R_\tau = R_\tau$;
2) $\|R_\tau\| = 1$ for every $\tau \in (0,1)$;
3) if $1 < p < \infty$ then R_τ and R_τ^* tend weakly to zero as $\tau \to 0$.

Theorem 4.8. *Let K_1 and K_2 be integral operators with homogeneous kernels over the ball B_n and $K_{1\tau}$ and $K_{2\tau}$ be the corresponding truncated operators. Then*

$$K_{1\tau}K_{2\tau} = K_\tau + P_\tau T P_\tau + R_\tau L R_\tau,$$

where K is also an operator with a homogeneous kernel and T and L are compact operators in $L_p(B_n)$.

The proof can be obtained by direct arguments.

Corollary *The commutator of $K_{1\tau}$ and $K_{2\tau}$ satisfies to equality*

$$[K_{1\tau}, K_{2\tau}] := K_{1\tau}K_{2\tau} - K_{2\tau}K_{1\tau} = P_\tau T P_\tau + R_\tau L R_\tau,$$

where T and L are compact operators on $L_p(B_n)$.

4.3. Pseudospectra

From now on we work in $L_2(B_n)$-space only. Our aim is to establish the relationship between spectral properties of the truncated operator K_τ and spectral properties of the original operator K. The notion of pseudospectrum for Wiener-Hopf operators was studied by Böttcher [1], see also Böttcher and Wolf [1].

Definition 4.9. *Let $\{E_\tau\}_{\tau\in(0,1)}$ be a family of sets $E_\tau \subset \mathbb{C}$. We denote by $\lim_{\tau\to 0} E_\tau$ the set of all $\lambda \in \mathbb{C}$ for which there are τ_1, τ_2, ... and λ_1, λ_2, ... such that*

$$\tau_1 > \tau_2 > \ldots > \tau_n \to 0, \quad \lambda_n \in E_{\tau_n}, \quad \lambda_n \to \lambda.$$

Definition 4.10. *For $\varepsilon > 0$, ε-pseudospectrum of an operator A is the set*

$$\Lambda_\varepsilon(A) = \left\{\lambda \in \mathbb{C} \, : \, \left\|(A - \lambda I)^{-1}\right\| \ge 1/\varepsilon\right\}.$$

It is known that the inclusion

$$\Lambda_\varepsilon(K) \subset \lim_{\tau\to 0} \Lambda_\varepsilon(K_\tau) \tag{4.7}$$

holds for every $\varepsilon > 0$. (In the general form this proposition was established in Böttcher [1].

The main result on the pseudospectra is given by the following theorem.

Theorem 4.11. *For each* $\varepsilon > 0$, *the relation is valid*

$$\lim_{\tau\to 0} \Lambda_\varepsilon(K_\tau) = \Lambda_\varepsilon(K).$$

The following theorem contains the conditions on the kernel $k(x, y)$ which guarantee the convergence of the usual spectra of the truncated operator K_τ to the spectra $\Omega_0(K)$ of the limiting operator K.

Theorem 4.12. *Let* $k(x, y)$ *satisfy the conditions* 1^0–3^0 *and: 1)* $k(x, y) = a(|x|, |y|)b(x' \cdot y')$; *2)* $a(1, \rho) = \overline{a(\rho, 1)}$; *3)* $b(t) = \overline{b(t)}$. *Then* $\lim\limits_{\tau\to 0} \Lambda_0(K_\tau) = \Lambda_0(K)$.

For the proofs of the results of this Section we refer to Avsyankin and Karapetiants [1]-[2].

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