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To the memory of Anatolyi P. Prudnikov

On some index relations for Bessel functions

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Abstract

The formula of the type $z^m\Big[K_{\nu+m}(z)-K_{\nu-m}(z)\Big]=\sum_{j=0}^{m-1}c_jz^jK_{\nu+j}(z)$ with exactly calculated coefficients c_j is proved together with similar formulas for $J_\nu(z)$ and $I_\nu(z)$. An application to the Bessel kernels $G_\alpha(\xi)$ is given.

Key Words and Phrases: special functions, Bessel function, modified Bessel function, McDonald function, recurrence relations, Bessel kernel, Riesz potential operator

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Let

$$J_\nu(z)=\sum_{k=0}^{\infty}(-1)^k\frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}\quad \text{and} \quad I_\nu(z)=\sum_{k=0}^{\infty}\frac{(z/2)^{2k+\nu}}{k!\Gamma(k+\nu+1)}$$

be the well known (see e.g. [1]-[3]) Bessel and modified Bessel function and

$$K_\nu(z)=\frac{\pi}{\sin\nu\pi}\left[I_{-\nu}(z)-I_\nu(z)\right],\nu\neq 0,\pm 1,...$$

the McDonald function (defined as $K_N(z)\ =\ \lim_{\nu\to N}K_\nu(z)\ ,\quad N=0,\pm 1,\cdots$ in case of integer values of the index).

The recurrence relations

$$z\Big[J_{\nu-1}(z)+J_{\nu+1}(z)\Big]=2\nu J_\nu(z)\ , \qquad z\Big[I_{\nu-1}(z)-I_{\nu+1}(z)\Big]=2\nu I_\nu(z)\ , \tag{1}$$

$$z\Big[K_{\nu+1}(z)-K_{\nu-1}(z)\Big]=2\nu K_\nu(z) \tag{2}$$

are well known, see e.g. [1], N 8.471.1, N 8.486.1 and N 8.486.10.

The aim of this note is to give an exact generalization of these relations to the case of the shifts $\nu \pm m$ of the index ν.

The formulas (3)-(6)given below are direct generalizations of (1)-(2), and it seems that they should be known. However, we failed to find anything of such a kind in the literature we know on special functions. We arrived at a necessity to use such relations in some inversion problems for potential type operators, see [4].

Theorem. *Let $\nu \in \mathbf{C}$. Then for any $m = 1, 2, 3, \ldots$ the relations hold*

$$z^m\Big[K_{\nu+m}(z)-K_{\nu-m}(z)\Big]=\sum_{j=0}^{m-1}(-1)^{m-j-1}b_m(j)z^jK_{\nu+j}(z)\ , \tag{3}$$

$$z^m\Big[I_{\nu-m}(z)-I_{\nu+m}(z)\Big]=\sum_{j=0}^{m-1}b_m(j)z^jI_{\nu+j}(z)\ , \tag{4}$$

$$z^m\Big[J_{\nu-m}(z)-(-1)^mJ_{\nu-m}(z)\Big]=\sum_{j=0}^{m-1}(-1)^jb_m(j)z^jJ_{\nu+j}(z)\ , \tag{5}$$

where

$$b_m(j)=2^{m-j}\binom{m}{j}\nu(\nu-1)\ldots(\nu+j-m+1)=\Gamma(\nu+1)\frac{2^{m-j}\binom{m}{j}}{\Gamma(\nu+j-m+1)}\ . \tag{6}$$

Proof. The relation (3) with the coefficients (6) is valid in the case $m = 1$ by (2). For $m > 1$ it will be proved by induction. But firstly, we pass to the case $m = 2$ to explain how we arrived at the expression (6) for the coefficients. To iterate the formula (2), we write

$$z^2\left[K_{\nu+2}(z)-K_{\nu+2}(z)\right]=z^2\left[K_{\nu+2}(z)-K_\nu(z)\right]+z^2\left[K_\nu(z)-K_{\nu-2}(z)\right]\ . \tag{7}$$

We apply (2) with ν replaced by $\nu+1$ in the first brackets in the right-hand side and with ν replaced by $\nu-1$ in the second ones, and get

$$z^2\left[K_{\nu+2}(z)-K_{\nu+2}(z)\right]=2z\left[(\nu+1)K_{\nu+1}(z)+(\nu-1)K_{\nu-1}(z)\right]\ .$$

Substituting here $zK_{\nu-1}(z)$ by its expression from (2) (with ν itself this time!), we arrive at

$$z^2\left[K_{\nu+2}(z)-K_{\nu-2}(z)\right]=4\nu zK_{\nu+1}(z)-2^2\nu(\nu-1)K_\nu(z)$$

which is nothing else but (3) with the coefficients (6) in the case $m = 2$.

Now we suppose that (3) is valid for some number m and all values of ν. Considering the left-hand side of (3) of order $m+1$, we represent it similarly to (7) as

$$z^{m+1}\Big[K_{\nu+m+1}(z)-K_{\nu-m-1}(z)\Big] = z^m\cdot z\Big[K_{\nu+m+1}(z)-K_{\nu+m-1}(z)\Big]+z\cdot z^m\Big[K_{\nu-1+m}(z)-K_{\nu-1-m}(z)\Big] \tag{8}$$

Since the formula (3) with the coefficients (6) is assumed to be valid for all ν, we may use this formula in the second term in the right-hand side of (8) with ν replaced by $\nu-1$, while the first term may be treated by the formula (2), with ν replaced by $\nu+m$. As a result we obtain

$$z^{m+1}\Big[K_{\nu+m+1}(z) - K_{\nu-m-1}(z)\Big] = 2(\nu+m)z^m K_{\nu+m}(z) + r\sum_{j=0}^{m-1}(-1)^{m-j-1}d_j z^j K_{\nu-1+j}(z) \ ,$$

where we have denoted

$$d_j = 2^{m-j}\binom{m}{j}(\nu-1)(\nu-2)\cdots(\nu+j-m) = \frac{\nu+j-m}{\nu}b_m(j)$$

for brevity. Now we lift the order of the McDonald functions in every term in the sum $\sum_{j=0}^{m-1}$ by means of the formula $zK_{\nu-1}(z) = -2\nu K_\nu(z) + rK_{\nu+1}(z)$ which is the same as (2). We obtain

$$z^{m+1}\Big[K_{\nu+m+1}(z) - K_{\nu-m-1}(z)\Big] = 2(\nu+m)z^m K_{\nu+m}(z)$$
$$+2\sum_{j=0}^{m-1}(-1)^{m-j}(\nu+j)d_j z^j K_{\nu+j}(z) + \sum_{j=0}^{m-1}(-1)^{m-j-1}d_j z^{j+1}K_{\nu+j+1}(z) \ .$$

We change the summation index j by $j-1$ in the second of these sums and note that

$$2(\nu+j)d_j + d_{j-1} = b_{m+1}(j)$$

which is checked directly. Hence

$$z^{m+1}\Big[K_{\nu+m+1}(z) - K_{\nu-m-1}(z)\Big] = 2^{m+1}(-1)^m\nu(\nu-1)...(\nu-m)K_\nu(z)$$
$$+\sum_{j=1}^{m-1}(-1)^{m-j}b_{m+1}(j)z^j K_{\nu+j}(z) + 2\nu(m+1)z^m K_{\nu+m}(z) = \sum_{j=1}^{m}(-1)^{m-j}b_{m+1}(j)z^j K_{\nu+j}(z)$$

which is the right-hand side of (3) exactly for the order $m+1$. The formula (3) has been proved.

The proof of the formula (4) follows exactly the same lines and therefore is omitted. The formula (5) is an immediate consequence of (4) because of the relation $I_\nu(z) = e^{-\frac{\pi i\nu}{2}}J_\nu(iz)$ between the Bessel and modified Bessel functions.

Let

$$G_\alpha(x) \ = \ \frac{2^{1-\frac{\alpha+n}{2}}}{\pi^{\frac{n}{2}}\Gamma(\frac{\alpha}{2})}\frac{K_{\frac{n-\alpha}{2}}(|x|)}{|x|^{\frac{n-\alpha}{2}}} \tag{7}$$

be the Bessel kernel well known in the theory of function spaces of fractional smoothness, see e.g. [5], Section 27, being the Fourier transformation of the function $(1+|x|^2)^{-\frac{\alpha}{2}}$:

$$(2\pi)^n G_\alpha(\xi) = \int_{R^n} \frac{e^{ix\xi}}{(1+|x|^2)^{\frac{\alpha}{2}}} dx \ .$$

Corollary. *Let $-n < \Re\alpha < n+2m$. Then the following relation holds for the Bessel kernels*

$$\sum_{k=0}^{m} (-1)^k a_{m,k}(\alpha) G_{n+\alpha+2k}(\xi) = (-1)^m a_m(\alpha) |\xi|^\alpha G_{n+2m-\alpha}(\xi) \ , \tag{10}$$

where

$$a_{m,k}(\alpha) = \alpha(\alpha-2)\cdots(\alpha-2m+2k+2)(n+\alpha(n+\alpha+2)\cdots(n+\alpha+2k-2)\binom{m}{k} =$$

$$2^m \binom{m}{k} \frac{\Gamma\left(\frac{\alpha}{2}+1\right)\Gamma\left(\frac{n+\alpha}{2}+k\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)\Gamma\left(\frac{\alpha}{2}-m+k+1\right)}$$

and

$$a_m(\alpha) = 2^{m-\alpha} \frac{\Gamma\left(\frac{n-\alpha}{2}+m\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \ .$$

In particular, in the case $m=1$ we have

$$\alpha G_{n+\alpha}(\xi) - (n+\alpha) G_{n+\alpha+2}(\xi) = -\frac{2^{1-\alpha}\Gamma\left(\frac{n-\alpha}{2}+1\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} |\xi|^\alpha G_{n+2-\alpha}(\xi) \ .$$

To get (10) from the formula (3), we choose $\nu = \frac{\alpha}{2}$ in (3) and calculate $z^j K_{\nu+j}(z) = z^{-\frac{\alpha}{2}} z^{\frac{\alpha}{2}+j} K_{\frac{\alpha}{2}+j}(z)$ in terms of $G_{n+\alpha+2j}$ according to (9) in the left-hand side of (3); as for the right-hand side of (3), we replace similarly $z^m K_{\nu-m}(z) = z^{\frac{\alpha}{2}} z^{m-\frac{\alpha}{2}} K_{m-\frac{\alpha}{2}}(z)$. After easy calculation of constants we arrive at (10).

Remark. *The relation of the type (10) is of importance in application to inversion problems for the Riesz potential operator, see [4].*

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