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UPPER AND LOWER BOUNDS FOR SOLUTIONS OF NONLINEAR VOLTERRA CONVOLUTION INTEGRAL EQUATIONS WITH POWER NONLINEARITY

Nikolai K.Karapetiants, Anatoly A.Kilbas, Megumi Saigo and Stefan G.Samko

Department of Mathematics, Rostov State University,
Rostov-on-Don 344104, Russia

Department of Mathematics and Mechanics, Belarusian State University,
Minsk 220050, Belarus

Department of Applied Mathematics, Fukuoka University,
Fukuoka 814-80, Japan

Unidade de Ciencias Exactas e Humanas, Universidade do Algarve,
Faro 8000, Portugal

Abstract

The Volterra nonlinear integral equation

$$\varphi^m(x) = a(x)\int_0^x k(x-t)b(t)\varphi(t)dt + f(x),\ 0 < x < d \leq \infty,$$

with $m > 1$ and real nonnegative functions $a(x)$, $k(u)$, $b(t)$ and $f(x)$ is studied. In the general case some upper bounds are given for the averages $\frac{1}{x}\int_0^x \varphi(t)dt$. In the case when $a(x)$, $k(u)$, $b(t)$ and $f(x)$ have power lower estimates near the origin, lower power type bounds directly for solutions $\varphi(x)$ are given. Conditions for the uniqueness of the solution in weighted space of continuous functions are proved. Particular cases of the above equation are specially considered.

KEY WORDS: Nonlinear Volterra integral equation, weighted space of continuous functions, uniqueness theorem

MSC (1991): 45D05, 45G10

1. Introduction.

We consider the Volterra nonlinear integral equation of the form

$$\varphi^m(x) = a(x) \int_0^x k(x-t) b(t) \varphi(t) dt + f(x),\ 0 < x < d \leq \infty, \tag{1.1}$$

with $m > 0$ and real-valued functions $a(x)$, $k(u)$, $b(t)$ and $f(x)$. This equation generalizes equations investigated by many authors. The equation

$$\varphi^m(x) = \int_0^x k(x-t) \varphi(t) dt + f(x),\ 0 < x < d \leq \infty, \tag{1.2}$$

arising in applications, e.g. in water perlocation [9], [23], [24] and in the nonlinear theory of wave propagation [13], was studied in [1], [4], [6], [10], [22], [23], [24], while more general equation

$$\varphi^m(x) = a(x) \int_0^x k(x-t) \varphi(t) dt + f(x),\ 0 < x < d \leq \infty, \tag{1.3}$$

- in [2], [3], [5], [7]. When $m > 1$, the equations (1.2) and (1.3) with $f(x) = 0$ may have a nontrivial solution $\varphi(x)$, see for example [22],[26]. All the papers above were devoted to investigation of problems concerning in main the existence and uniqueness of a solution $\varphi(x)$ for equations (1.2) and (1.3) with $m > 1$, in some spaces of continuous or integrable functions. The equation (1.2) with $0 < m < 1$ and a continuous kernel $k(u)$ was considered in [1], [4], where some results were given on the uniqueness of its solution $\varphi(x)$ in some spaces of continuous or integrable functions. Such a problem for the equation (1.2) with $m < 0$ and non-increasing kernel $k(u)$ in the class of almost decreasing functions was studied in [12]. Lower estimates and asymptotic properties near zero for the solution $\varphi(x)$ of the equation (1.3) with $m > 1$ were obtained in [11] provided that $a(x)$, $k(u)$ and $f(x)$ have power asymptotic behavior near zero.

The existence of the solution for the equation

$$\varphi^m(x) = a(x) \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt + f(x),\ 0 < x < d \leq \infty, \tag{1.4}$$

with $m > 0$ and weakly singular kernel $k(u) = u^{\alpha-1}$ for $0 < \alpha < 1$ was investigated in spaces of locally integrable and continuous functions in [14], [15], [19]. Asymptotic properties at zero of the solution $\varphi(x)$ for the equation (1.4) with $m \in R = (-\infty, \infty)$, $m \neq 0, -1, -2, \cdots$, in the case when $a(x)$ and $f(x)$ have special asymptotic at zero were studied in [16], [17], [18], [20], [25]. Special cases of (1.4), when its solution $\varphi(x)$ can be found in closed form, were investigated in [11], [18], [19], [20] and [26].

The equation of the form

$$\varphi^m(x) = a(x) \int_0^x b(t) \varphi(t) dt + f(x),\ 0 < x < d \leq \infty, \tag{1.5}$$

with real m was considered in [21], where existence and uniqueness results were discussed and special cases of solution in closed form were treated.

The main results in this paper are lower bounds given in Theorem 2.1, upper bounds for averages of solutions given in Theorem 4.1 and uniquness theorem (see Theorem 5.1). These results are obtained in the case of $m > 1$ and real nonnegative functions $a(x)$, $k(u)$, $b(t)$ and $f(x)$.

2. Lower estimates for a solution of integral equation (1.1)

Let $C(0,d),\ 0<d\le\infty$, be the space of real valued continuous functions on $(0,d)$, and let $L_1^{loc}(0,d)$ be the space of all Lebesgue measurable functions which are in $L_1(0,d_0)$ for all d_0 such that $0<d_0<d$. We denote by $CL_{loc}(0,d)$ the intersection of $C(0,d)$ and $L_1^{loc}(0,d)$:

$$CL_{loc}(0,d)=C(0,d)\bigcap L_1^{loc}(0,d), \tag{2.1}$$

so that a function in $CL_{loc}(0,d)$ may have singularities only at the end points of $(0,d)$. By $CL_{loc}^+(0,d)$ we denote the nonnegative functions in $CL_{loc}(0,d)$.

Let $C[0,d)$ be the space of continuous functions on $[0,d)$. Let $b(x)$ be a nonnegative function on $[0,d)$. We denote by $C([0,d),b)$ the space of functions $g(x)$ such that $b(x)g(x)\in C[0,d)$, and by $C^+[0,d)$ and $C^+([0,d),b)$ - the subclasses of $C[0,d)$ and $C([0,d),b)$, respectively, which contain nonnegative functions. Similarly, $CL_{loc}((0,d),b))$ and $CL_{loc}^+((0,d),b))$ are subclasses of functions $g(x)$ such that $b(x)g(x)\in CL_{loc}(0,d)$ or $b(x)g(x)\in CL_{loc}^+(0,d)$, respectively.

Obviously,

$$C([0,d),b)\subset CL_{loc}((0,d),b),\ \ C^+([0,d),b)\subset CL_{loc}^+((0,d),b). \tag{2.2}$$

Remark 2.1. *It is clear that the equation (1.1) in case $b(x)\not\equiv 0$ in a neighborhood of the origin, may be reduced to the equation only with one coefficient:*

$$\varphi_1^m(x)=a_1(x)\int_0^x k(x-t)\varphi_1(t)dt+f_1(x),\ \ 0<x<d\le\infty, \tag{2.3}$$

where $\varphi_1(x)=b(x)\varphi(x), a_1(x)=[b(x)]^m a(x)$ and $f_1(x)=[b(x)]^m f(x)$ with the solutions $\varphi_1(x)$ looked for in the space $CL_{loc}^+(0,d)$. However we find it more convenient to consider the equations just in the form (1.1).

In this section we give an apriori lower estimate for a nonnegative solution $\varphi(x)$ of the equation (1.1) with $m>1$, under the assumption that the functions $a(x),\ k(x),\ b(x),\ f(x)\in CL_{loc}^+(0,d)$ have lower power bounds, see Theorem 2.1 below. We assume that solutions of equation (1.1) are in the space $CL_{loc}^+([0,d),b)$. Since $CL_{loc}^+([0,\infty))$ is a ring with respect to Volterra convolution (see, [11], Theorem 1), the integral term in (1.1) is also a locally integrable function. Therefore, the equation (1.1) is well posed in this class.

We note that the condition

$$f^{\frac{1}{m}}(x)\in CL_{loc}^+([0,d),b) \tag{2.4}$$

is necessary for solvability of equation (1.1) in $CL_{loc}^+([0,d),b)$. This fact is a consequence of the evident inequality $f(x)\le\varphi^m(x)$ and the assumptions on the solution $\varphi(x)$. We also suppose that $f(x)$ satisfies the condition

$$f(x)\not\equiv 0,\ x\in(0,\varepsilon)\quad\text{for all}\quad \varepsilon>0. \tag{2.5}$$

Otherwise, if $f(x)\equiv 0,\ x\in(0,\varepsilon_0)$ with some $\varepsilon_0<d$ and $f(x)\not\equiv 0,\ x\in(\varepsilon_0,\varepsilon_0+\varepsilon)$ for all $\varepsilon>0$ we may pass to the function

$$\psi(x)=\begin{cases}0, & 0<x<\varepsilon_0,\\ \varphi(x+\varepsilon_0), & \varepsilon_0<x<d-\varepsilon_0\end{cases} \tag{2.6}$$

and obtain the same equation (1.1) with respect to $\psi(x)$, for which the condition (2.5) is satisfied.

In the case when $f(x) \equiv 0,\ x \in (0,d)$, the equation (1.1) also may be investigated, but in this case we need some special additional assumptions on the functions $a(x),\ k(x),\ b(x)$. We consider this case specially in Section 3. But before that in this section we suppose that $f(x)$ satisfies the conditions (2.5).

Theorem 2.1. *Let $m,\ \alpha,\ \mu,\ \nu \in \mathbf{R}$ be such that*

$$m > 1,\ \ 0 < \alpha \leq 1,\ \ \mu + \alpha + m\nu > 1, \tag{2.7}$$

and let nonnegative coefficients $a(x),\ b(x)$ and nonnegative kernel $k(x) \in CL^+_{loc}(0,d)$ satisfy the conditions

$$a(x) \geq ax^{\mu},\ \ k(x) \geq kx^{\alpha-1},\ \ b(x) \geq bx^{\nu-1} \tag{2.8}$$

with $a > 0,\ k > 0,\ b > 0$. Let also $f(x)$ satisfy the conditions (2.4)-(2.5). If equation (1.1) is solvable in $CL^+_{loc}((0,d),b)$, then its solution $\varphi(x)$ satisfies the estimate

$$\varphi(x) \geq Ax^{(\mu+\alpha+\nu-1)/(m-1)}, \tag{2.9}$$

where

$$A = \left[akbB\left(\alpha, \frac{\mu+\alpha+m\nu-1}{m-1}\right)\right]^{1/(m-1)} \tag{2.10}$$

and $B(z,w)$ is the Euler beta function. The constant A is precise in the sense that we have exact equality in (2.9) when $f(x) \equiv 0$ and inequalities in (2.8) are replaced by equalities.

Proof. According to (2.8) and (1.1) we have

$$\varphi^m(x) \geq akbx^{\mu} \int_0^x \frac{t^{\nu-1}}{(x-t)^{1-\alpha}} \varphi(t)dt \tag{2.11}$$

and since $0 < \alpha \leq 1$, we obtain

$$\varphi^m(x) \geq akbx^{\mu+\alpha-1} \int_0^x t^{\nu-1}\varphi(t)dt. \tag{2.12}$$

We denote

$$y(x) = \int_0^x t^{\nu-1}\varphi(t)dt. \tag{2.13}$$

Since $\varphi(x) \in CL^+_{loc}((0,d),b)$, then $\varphi(x) \in CL^+_{loc}((0,d),t^{\nu-1})$ so the integral on the right hand-side of (2.13) exists and $y'(x) \in C(0,d)$ with $y(0) = 0$. Then $\varphi(x) = x^{1-\nu}y'(x)$, and we rewrite (2.12) in the form

$$[x^{1-\nu}y'(x)]^m \geq akbx^{\mu+\alpha-1}y(x).$$

Since $\varphi^m(x) \geq f(x)$, we have

$$y(x) = \int_0^x t^{\nu-1}\varphi(t)dt \geq \int_0^x t^{\nu-1}f^{\frac{1}{m}}(t)dt.$$

Therefore, $y(x) > 0$ in some neighborhood of the origin, by (2.5). Then from (2.12) it is seen that $\varphi(x) > 0$ for all $x \in (0, d)$ and then (2.13) implies that $y(x) > 0$ for all $x \in (0, d)$. Taking this into account, we obtain

$$y'(x)y^{-1/m}(x) \geq (akb)^{1/m} x^{[(\mu+\alpha-1)/m]+\nu-1}. \tag{2.14}$$

Integrating the latter inequality over $(0, x)$ and using that $y(0) = 0$ and the conditions in (2.7), we arrive at the estimate

$$y(x) \geq (akb)^{1/(m-1)} \left[\frac{m-1}{\mu+\alpha+m\nu-1}\right]^{m/(m-1)} x^{(\mu+\alpha+m\nu-1)/(m-1)}. \tag{2.15}$$

Then from (2.12) and (2.13), the estimate

$$\varphi(x) \geq D x^{(\mu+\alpha+\nu-1)/(m-1)} \tag{2.16}$$

easily follows, but with the constant

$$D = \left[\frac{akb(m-1)}{\mu+\alpha+m\nu-1}\right]^{1/(m-1)}.$$

It remains to improve the constant. Using the estimate (2.16) on the right-hand side of (2.11), we obtain

$$\varphi^m(x) \geq DakbB\left(\alpha, \nu + \frac{\mu+\alpha+\nu-1}{m-1}\right) x^{(\mu+\alpha+\nu-1)m/(m-1)},$$

or

$$\varphi(x) \geq D^{1/m} A^{(m-1)/m} x^{(\mu+\alpha+\nu-1)/(m-1)}.$$

Substituting again this estimate into (2.11), we obtain in a similar way

$$\varphi(x) \geq D^{1/m^2} A^{(m-1)[1+1/m]/m} x^{(\mu+\alpha+\nu-1)/(m-1)}.$$

Repeating this operation n times, we find

$$\varphi(x) \geq D^{1/m^n} A^{(m-1)[1+1/m+\cdots+1/m^{n-1}]/m} x^{(\mu+\alpha+\nu-1)/(m-1)}.$$

Taking the limit as $n \to \infty$, we arrive at the inequality (2.9) with the required constant A.

Evidently $A \geq D$, since $B(z, w) > \frac{1}{z}$ for $z > 0$ and $0 < w \leq 1$. The preciseness of the constant A may be checked by direct verification of the fact that the function $Ax^{(\mu+\alpha+\nu-1)/(m-1)}$ satisfies the equation (1.1) when $f(x) \equiv 0$ and $a(x) = ax^{\mu}$, $k(x) = kx^{\alpha-1}$, $b(x) = bx^{\nu-1}$.

Corollary 2.1. *Let $m > 1$, $0 < \alpha \leq 1$ and μ, $\nu \in \mathbf{R}$ be such that $\mu + \alpha + m\nu > 1$, $a > 0$ and let the function $f^{\frac{1}{m}}(x) \in CL^{+}_{loc}((0, d), t^{\nu-1})$ and satisfy the condition (2.5). If the equation*

$$\varphi^m(x) = ax^{\mu} \int_0^x \frac{t^{\nu-1}}{(x-t)^{1-\alpha}} \varphi(t)dt + f(x), \ 0 < x < d \leq \infty, \tag{2.17}$$

is solvable in $CL^+_{loc}((0,d),t^{\nu-1})$*, then its solution* $\varphi(x)$ *satisfies the estimate*

$$\varphi(x) \geq A_1 x^{(\mu+\alpha+\nu-1)/(m-1)} \tag{2.18}$$

with $A_1 = \left[aB\left(\alpha, \frac{\mu+\alpha+m\nu-1}{m-1}\right)\right]^{1/(m-1)}$.

We note that the condition $f^{\frac{1}{m}}(x) \in CL^+_{loc}((0,d),t^{\nu-1})$ of Corollary 2.1 is satisfied if, for example, $f(x) \in CL^+_{loc}(0,d)$ and $\nu m > 1$.

Corollary 2.2. *Under the assumptions of Theorem 2.1, if equation (1.1) has a solution in* $CL^+_{loc}((0,d),b)$ *with the asymptotic behavior*

$$\varphi(x) = cx^\gamma + o\left(x^\gamma\right), \ c \neq 0, \tag{2.19}$$

as $x \to 0$*, then necessarily*

$$\gamma \leq \frac{\mu+\alpha+\nu-1}{m-1}. \tag{2.20}$$

Corollary 2.3. *Let* $m > 1$*,* $\mu, \ \nu \in \mathbf{R}$ *be such that* $\mu + m\nu > 0$*, and let nonnegative coeeficients* $a(x)$*,* $b(x)$ *satisfy the conditions* $a(x) \geq ax^\mu$*,* $b(x) \geq bx^{\nu-1}$ *with* $a > 0$ *and* $b > 0$*. Let also* $f^{\frac{1}{m}}(x) \in CL^+_{loc}((0,d),b)$ *and satisfy the condition (2.5). If equation (1.5) is solvable in* $CL^+_{loc}((0,d),b)$*, then its solution* $\varphi(x)$ *satisfies the estimate*

$$\varphi(x) \geq A_2 x^{(\mu+\nu)/(m-1)}, \ A_2 = \left[\frac{ab(m-1)}{\mu+m\nu}\right]^{1/(m-1)}. \tag{2.21}$$

Remark 2.2. *By (2.2), the statements of Theorem 2.1 and Corollary 2.1 are also valid for continuous solutions in the weighted space* $C([0,d),b)$*.*

3. The equation (1.2) in the case $f(x) \equiv 0$.

By $\mathbf{CL}^+_{loc}(0,d)$ and $\mathbf{C}^+[0,d)$ we denote the subclass of functions $\varphi(x) \in CL^+_{loc}(0,d)$ or $\varphi(x) \in C^+[0,d)$, respectively, such that $\varphi(x) > 0$ for $x > 0$. Theorem 2.1 on lower estimates of solutions of equation (1.2) is also valid in the case $f(x) \equiv 0, \ x \in (0,d)$, that is, the equation

$$\varphi^m(x) = \int_0^x k(x-t)\varphi(t)dt, \ 0 < x < d \leq \infty, \tag{3.1}$$

if we look apriori for solutions in the subclass $\mathbf{CL}^+_{loc}(0,d)$.

Theorem 3.1. *Let* $m > 1$ *and let* $k(x) \in CL^+_{loc}(0,d)$ *satisfy the condition*

$$k(x) \geq kx^{\alpha-1} \tag{3.2}$$

with $k > 0$ *and* $0 < \alpha \leq 1$*. If equation (3.1) is solvable in* $\boldsymbol{CL}^+_{loc}(0,d)$*, then its solution* $\varphi(x)$ *satisfies the estimate (2.9) with the constant* A *in (2.10) calculated for* $\mu = 0, \ \nu = 1$*.*

The proof of Theorem 3.1 is in fact the same as that of Theorem 2.1, if we take into account that the condition (2.5) on $f(x)$ was used only to show that $\varphi(x) > 0$ for $x > 0$. Now we have this by definition of the class $\mathbf{CL}^+_{loc}(0,d)$.

The assumption that the solution $\varphi(x)$ is positive for $x > 0$ is natural, which is seen from the following lemma.

Lemma 3.1. *Let $m > 1$ and let $k(x) \in L^+_{loc}(0, d)$ be non-zero in a neighborhood of the origin:*

$$k(x) \not\equiv 0,\ x \in (0, \delta) \text{ for all } \delta > 0. \tag{3.3}$$

If equation (3.1) is solvable in $CL^+_{loc}(0, d)$, then for the solution $\varphi(x)$ only one of the following cases may realize:
1) $\varphi(x) \equiv 0, 0 \le x \le d$;
2) $\varphi(x) > 0, 0 < x \le d$;
3) there exits $d_0 \in (0, d)$ such that $\varphi(x) \equiv 0, 0 \le x \le d_0$ and $\varphi(x) > 0, d_0 < x \le d$.

Proof. Suppose that equation (3.1) has a solution in $CL^+_{loc}(0, d)$ such that $\varphi(x_0) > 0$. Then $\varphi(x) > 0$ for all $x_0 \le x \le d$. Indeed, we have $\varphi(t) > \frac{\varphi(x_0)}{2}$ for $t \in (x_0, x_0 + \delta)$ for sufficiently small δ. Then

$$\varphi^m(x) = \int_0^x k(x-t)\varphi(t)dt \ge \int_{x_0}^{x_0+\delta} k(x-t)\varphi(t)dt \ge \frac{\varphi(x_0)}{2}\int_0^\delta k(t)dt > 0. \tag{3.4}$$

This gives us the cases 1)-3). We observe that the case 3) may be reduced to the case 2) by the passing to the function $\psi(x)$ from (2.6). Lemma is proved.

Remark 3.1. *If $k(x)$ in conditions of Lemma 3.1 instead the condition (3.3) satisfy the similar condition, namely $k(x) \equiv 0,\quad x \in (0, \delta_0)$ and $k(x) \not\equiv 0,\quad x \in (\delta_0, \delta_0 + \delta)$ for all $\delta > 0$ we can repeat our discussion with passing to the function $\psi(x)$ and obtain the result as in Lemma 3.1 with shift on δ_0.*

4. The upper and lower estimates for averages of solutions.

In the case of the equation (1.2) we intend can obtain the upper bounds for the "averages"

$$\frac{1}{x}\int_0^x \varphi(t)dt$$

when m is an integer: $m = 2, 3, 4, ...$ To this end, we need the following auxiliary lemma.

Lemma 4.1. *Let $a \ge 0, b > 0$ and $m > 0$. The equation $\xi^m = a\xi + b, \xi > 0$, has a unique solution x_0, so that*

$$\xi^m \le a\xi + b \iff 0 < \xi \le \xi_0 \quad \text{and} \quad \xi^m \ge a\xi + b \iff \xi \ge \xi_0. \tag{125.000}$$

In the case $m = 2, 3, ...$, the following conclusion is also valid

$$\xi^m \le a\xi + b \implies \xi < a^{\frac{1}{m-1}} + b^{\frac{1}{m}}.$$

Proof. The first statement of Lemma is obvious. The proof of the second part is motivated by the first one. Let $g(\xi) = \xi^m - a\xi - b$. By means of the binomial formula, it is easy to show that

$$g(a^{\frac{1}{m-1}} + b^{\frac{1}{m}}) > 0. \tag{4.1}$$

Then by the first part it should be $0 < \xi \leq \xi_1 \leq a^{\frac{1}{m-1}} + b^{\frac{1}{m}}$. However, we may give the direct rigorous proof. Indeed, let $\xi^m \leq a\xi + b$ and, on the contrary, $\xi \geq a^{\frac{1}{m-1}} + b^{\frac{1}{m}}$. Then

$$a\xi + b \geq \xi^m \geq \left(a^{\frac{1}{m-1}} + b^{\frac{1}{m}}\right)^m = a^{\frac{m}{m-1}} + mab^{\frac{1}{m}} + ... + b \geq a(a^{\frac{1}{m-1}} + mb^{\frac{1}{m}}) + b.$$

Hence

$$\xi \geq a^{\frac{1}{m-1}} + mb^{\frac{1}{m}}. \tag{4.2}$$

Repeating these arguments n times, we find

$$\xi \geq a^{\frac{1}{m-1}} + m^n b^{\frac{1}{m}}. \tag{4.3}$$

for an arbitrary n, which is impossible. This completes the proof of the lemma.

Theorem 4.1. *Let $k(x),\ f(x) \in L^+_{loc}(0,d)$ and $m = 2, 3, ...$ If equation (1.2) is solvable in $CL^+_{loc}(0,d)$, then its solution $\varphi(x)$ admits the estimates*

$$\frac{1}{x}\int_0^x f^{\frac{1}{m}}(t)dt \leq \frac{1}{x}\int_0^x \varphi(t)dt \leq \left(\int_0^x k(t)dt\right)^{\frac{1}{m-1}} + \left(\frac{1}{x}\int_0^x f(t)dt\right)^{\frac{1}{m}}. \tag{4.4}$$

Proof. The left-hand inequality in (4.4) is obvious, since $\varphi^m(x) \geq f(x)$. To prove the right-hand one, we integrate equation (1.2) and obtain

$$\int_0^x \varphi^m(t)dt \leq K(x)\Phi(x) + F(x). \tag{4.5}$$

where $\Phi(x) = \int_0^x \varphi(t)dt$, $K(x) = \int_0^x k(t)dt$ and $F(x) = \int_0^x f(t)dt$. Using the Hölder inequality, we obtain

$$x^{m-1}\int_0^x \varphi^m(t)dt \geq \left(\int_0^x \varphi(t)dt\right)^m, \tag{4.6}$$

so that

$$\Phi^m(x) \leq x^{m-1}K(x)\Phi(x) + x^{m-1}F(x). \tag{4.7}$$

Then the right-hand inequality in (4.4) follows from (4.7) by Lemma 4.1, where one should take $\beta = \Phi(x), a = x^{m-1}K(x)$ and $b = x^{m-1}F(x)$. The theorem has been proved.

Hypothesis. *Under the assumptions of Theorem 4.1 on $k(x)$ and $f(x)$, the estimate (4.4) is probably valid for all $m > 1$.*

Corollary 4.1. *Under the assumptions of Theorem 4.1, if $\varphi(x)$ is a solution of the equation (1.2), then*

$$\int_0^x \varphi(t)dt = o(x^{1-\frac{1}{m}}), \quad x \to 0.$$

Under the assumption that the solution is bounded, we can obtain the upper estimate for the solution itself.

Theorem 4.2. *Let $k(x) \in L^+_{loc}(0,d)$, $f(x) \in L^+_{\infty}(0,d)$ and $m = 2, 3, ...$ If equation (1.2) is solvable in $L^+_{\infty}(0,d)$, then*

$$f^{\frac{1}{m}}(x) \leq \varphi(x) \leq \left(\int_0^x k(t)dt\right)^{\frac{1}{m-1}} + \left(\sup_{0\leq t\leq x} f(t)\right)^{\frac{1}{m}}. \tag{4.8}$$

Proof. To prove the right-hand side inequality in (4.8), we use the estimate

$$\left(\sup_{0\le t\le x}\varphi(t)\right)^m=\sup_{0\le t\le x}\varphi^m(t)\le\left(\sup_{0\le t\le x}\varphi(t)\right)\int_0^x k(t)dt+\sup_{0\le t\le x}f(t).$$

Applying Lemma 4.1 with the evident choice of β, a and b, we have

$$\sup_{0\le t\le x}\varphi(t)\le\left(\int_0^x k(t)dt\right)^{\frac{1}{m-1}}+\left(\sup_{0\le t\le x}f(t)\right)^{\frac{1}{m}}, \tag{4.9}$$

which yields (4.8). The theorem has been proved.

Remark 4.2. *Let $f(x)\equiv 0$ and $k(x)\in L^+_{loc}(0,d)$. If equation (1.2) is solvable in $L^+_\infty(0,d)$, then*

$$\varphi(x)\le\left(\int_0^x k(t)dt\right)^{\frac{1}{m-1}}. \tag{4.10}$$

for all $m>1$

A similar estimate was known before under the additional assumption that the kernel $k(t)$ is a continuous increasing function (see, for example, [22],[5])).

Theorem 4.3. *Let $k^s(x)\in L^+_{loc}(0,d)$ for some $s>1$ and $f(x)\in C^+[0,d)$. If equation (1.2) is solvable in $L^+_{loc}(0,d)$ then its solution $\varphi(x)$ belongs to $C^+[0,d)$*

Proof. The proof is similar to that in [5], where this theorem was given in the case $f(x)\equiv 0$. First of all we show that

$$\varphi^p(x)\in L^+_{loc}(0,d)\ \text{ for all }\ p\ge 1. \tag{4.11}$$

Indeed, from the equation (1.2) we see that the convolution on the right-hand side in (1.2) belongs to $L_s(0,d_0)$. Consequently, $\varphi(x)\in L_{ms}(0,d_0)$. Repeating our arguments n times, we obtain that $\varphi(x)\in L_{sm^n}(0,d_0)$ for all n, which yields (4.11). To prove the theorem, it remains to choose n such that $sm^n>s'$, $s'=\frac{s}{s-1}$ and take into account that $L_s(0,d_0)*L_{s'}(0,d_0)\subset C[0,d_0)$.

We note that the statement (4.11) of Theorem 4.3 holds also in the case $s=1$.

Corollary 4.2. *Let $m>1$, $a>0$, $\alpha>0$ and the function $f(x)\in C^+[0,d)$.*

If the integral equation

$$\varphi^m(x)=a\int_0^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt+f(x),\ 0<x<d\le\infty, \tag{4.12}$$

is solvable in $C^+[0,d)$, then its solution satisfies the estimate

$$f^{\frac{1}{m}}(x)\le\varphi(x)\le\left(\frac{a}{\alpha}\right)^{\frac{1}{m-1}}x^{\frac{\alpha}{m-1}}+\sup_{0\le t\le x}f^{\frac{1}{m}}(t). \tag{4.13}$$

5. Uniqueness of solution of integral equation (1.1)

In this section we apply Theorem 2.1 to give some conditions for the uniqueness of the solution $\varphi(x)$ of equation (1.1) in $C^+([0,d),b)$.

Theorem 5.1. *Let $m>1$, $0<\alpha\le 1$ and μ, $\nu\in\mathbf{R}$ be such that $\mu+\alpha+m\nu>1$. Let ℓ be any number such that $0<\ell<d$. Suppose that the functions $a(x)$ $b(x)\in C^+(0,d)$, $k(x)\in L^+_{loc}(0,d)$ and $f^{\frac{1}{m}}(x)\in C^+([0,d),t^{\nu-1})$ satisfy the conditions (2.8) and (2.5). Let*

$$M:=\sup_{x\in(0,\ell)} x^{1-\mu-\nu-\alpha}a(x)b(x)\int_0^x k(t)dt<mA^{m-1}, \tag{5.2}$$

where A is defined in (2.10). If equation (1.1) is solvable in $C^+([0,\ell),b)$, then its solution $\varphi(x)$ is unique on the interval $[0,\ell]$.

Proof. Let $\varphi_1(x)$ and $\varphi_2(x)$ be two solutions of equation (1.1) in $C^+([0,\ell),b)$. Since $m>1$, by mean value theorem we have

$$|[\varphi_1(x)]^m-[\varphi_2(x)]^m|\ge m|\varphi_1(x)-\varphi_2(x)|\left(\min[\varphi_1(x),\varphi_2(x)]\right)^{m-1}.$$

Hence, in accordance with the lower estimate (2.10) and Remark 2.2, we have

$$|[\varphi_1(x)]^m-[\varphi_2(x)]^m|\ge mA^{m-1}x^{\mu+\alpha+\nu-1}|\varphi_1(x)-\varphi_2(x)|.$$

Then by (1.1),

$$mA^{m-1}x^{\mu+\alpha+\nu-1}|\varphi_1(x)-\varphi_2(x)|\le a(x)\int_0^x k(x-t)b(t)|\varphi_1(t)-\varphi_2(t)|dt. \tag{5.3}$$

Let

$$u(x)=b(x)|\varphi_1(x)-\varphi_2(x)|. \tag{5.4}$$

Then (5.3) can be rewritten as

$$u(x)\le\frac{a(x)b(x)}{mA^{m-1}x^{\mu+\alpha+\nu-1}}\int_0^x k(x-t)u(t)dt. \tag{5.5}$$

Let ℓ_0 be a number arbitrarily close to $\ell, 0<\ell_0<\ell$ and x_0 be the maximum point of $u(x)$ on $[0,\ell_0]:\ u(x_0)=\max_{0\le x\le\ell_0}u(x)$. Then

$$\int_0^{x_0}k(x_0-t)u(t)dt\le K(x_0)u(x_0). \tag{5.6}$$

where $K(x)=\int_0^x k(t)dt$. Substituting this into (5.5) and using the notation in (5.2), we arrive at the estimate

$$u(x_0)\le\frac{M}{mA^{m-1}}u(x_0) \tag{5.7}$$

Since $M<mA^{m-1}$, we obtain $u(x_0)=0$ at the maximum point of $u(x)$ on an arbitrary subinterval $[0,\ell_0]$. Then $u(x)\equiv 0$, so that $\varphi_1(x)=\varphi_2(x)$, which proves the theorem.

Of course, the condition (5.2) is not necessary for uniqueness, which may be illustrated by the following equation

$$\varphi^2(x)=\frac{a}{x^\alpha}\int_0^x\frac{\varphi(t)}{(x-t)^{1-\alpha}}dt+f,\ 0<x<d, \tag{5.8}$$

where f is a constant, satisfying the condition $f \le (\frac{a}{2\alpha})^2$. This equation is solvable in $C^+[0,d)$ and has the unique solution $\varphi(x) = c$, where the constant c is the positive solution of the equation $\alpha c^2 - ac - \alpha f = 0$, but the condition (5.2) is violated turning into the equality $M = \frac{a}{\alpha} = 2A$.

Modifying this example, we note that the equation

$$\varphi^2(x) = \frac{a}{|x-1|^\alpha} \int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}} dt, \ 0 < x < 2,$$

has the discontinuous solution: $\varphi(x) = 0$, if $0 \le x \le 1$, and $\varphi = \frac{a}{\alpha}$, if $1 < x \le 2$. This example shows that Theorem 4.3 cannot not be valid in the general case of equation (1.1), if $a(x)$ has singularities beyond the origin.

Corollary 5.1. *Let $m > 1$, $a > 0$, $0 < \alpha \le 1$ and μ, $\nu \in \mathbf{R}$ be such that*

$$\mu + \alpha + m\nu > 1, \ \frac{a}{\alpha} < mB\left[\alpha, \frac{\mu + \alpha + m\nu - 1}{m-1}\right], \tag{5.9}$$

and let the function $f^{\frac{1}{m}}(x) \in C^+([0,d), t^{\nu-1})$ satisfy the condition (2.5).

If equation (2.17) is solvable in $C^+([0,d), x^{\nu-1})$, then its solution $\varphi(x)$ is unique.

Theorem 5.2. *Let $m > 1$ and μ, $\nu \in \mathbf{R}$ be such that $\mu + m\nu > 1$. Let the functions $a(x)$, $b(x) \in C^+(0,d)$ and $f^{\frac{1}{m}}(x) \in C^+([0,d), b)$ satisfy the conditions (2.8) and (2.5). Let*

$$M := \sup_{x \in (0,d)} x^{1-\mu-\nu} a(x) b(x) < ab \frac{m(m-1)}{m\nu + \mu}$$

and equation (1.5) is solvable in $C^+([0,d), b)$, then its solution $\varphi(x)$ is unique.

Corollary 5.2. *Let $a > 0$ and m, μ, $\nu \in \mathbf{R}$ be such that*

$$m > 1, \ 0 < \mu + m\nu < m(m-1), \tag{5.11}$$

and let the function $f^{\frac{1}{m}}(x) \in C^+([0,d), t^{\nu-1})$ satisfy the condition (2.4). If the integral equation

$$\varphi^m(x) = ax^\mu \int_0^x t^{\nu-1} \varphi(t) dt + f(x), \ 0 < x < d \le \infty. \tag{5.12}$$

is solvable in the space $C^+([0,d), x^{\nu-1})$, then its solution $\varphi(x)$ is unique.

In particular, Corollary 5.2 holds for equation (5.12) in the case $\mu = 0$ and $\nu = 1$ if $m > 2$.

Remark 5.1. *Corollary 5.2 gives some conditions easily verified when m is given and we want to know what values of μ and ν are admissible in (5.11) for the uniqueness of the solution. It is easy to check that, inversely for given values of μ and ν the possible values of m are described as follows.*

Given μ and ν, the conditions (5.11) are satisfied if

1) In the case $\mu > -\left(\frac{1+\nu}{2}\right)^2$,

$$\text{either } \ m > max\left(1, -\frac{\mu}{\nu}, \frac{1+\nu+\sqrt{D}}{2}\right) \quad \text{or } \ 1 < m < \frac{1+\nu-\sqrt{D}}{2}, \tag{5.13}$$

when $\nu > 0$, *and*

$$m > max\left(1, \frac{1+\nu+\sqrt{D}}{2}\right),$$

when $\nu < 0$; *here* $D = (1+\nu)^2 + 4\mu$ *and the interval* $1 < m < \frac{1+\nu-\sqrt{D}}{2}$ *in (5.13) is non-empty if and only if* $\nu > max(1, -\mu)$;

2) In the case $\mu \leq -\left(\frac{1+\nu}{2}\right)^2$,

$$m > max\left(1, -\frac{\mu}{\nu}\right) \quad if \quad \nu > 0 \tag{5.14}$$

$$1 < m < -\frac{\mu}{\nu} \quad if \quad \nu < 0 \tag{5.15}$$

with the value $m = \frac{1+\nu}{2}$ *excluded in (5.14)-(5.15) in the case* $\mu = -\left(\frac{1+\nu}{2}\right)^2$;

3) In the remaining case $\nu = 0$, *we have* $\mu > 0$ *and the following inequality for* m:

$$m > max\left(1, \frac{1+\sqrt{1+4\mu}}{2}\right).$$

In particular, in the case $\nu = 1$ *we have*

$$m > max\left(1, -\mu, 1+\sqrt{1+\mu}\right), \quad \text{if} \quad \mu > -1,$$

$$m > max(1, -\mu), \quad \text{if} \quad \mu \leq -1,$$

Remark 5.2. *If* $a(x) \neq 0$ *for* $x \in [0, d)$ *in the equation (1.5), then this equation is equivalent to the following Cauchy problem for the differential equation*

$$y'(x) = p(x)y^{\frac{1}{m}}(x) + q'(x),$$

$$y(0) = q(0),$$

where

$$y(x) = \frac{\varphi^m(x)}{a(x)}, \quad \alpha(x) = b(x)a^{\frac{1}{m}}(x), \quad \beta(x) = \frac{f(x)}{a(x)}$$

Thus, results similar to those in Theorems 5.2 are also valid for this problem.

6. The case of singular coefficient

We return here to the integral equation of the type (5.8) with the singular coefficient:

$$\varphi^m(x) = \frac{a}{x^\alpha}\int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt + f(x), \quad 0 < x < d, \tag{6.1}$$

We suppose that $f(x) \in C^+[0, d)$ and look for solutions in $L^+_{loc}(0, d)$. Theorem 4.3 states that in the case when the coefficient of the equation is not singular, any solution which apriori is in $L^+_{loc}(0, d)$, in fact is in $C^+[0, d)$. We show that this is also valid in the case of singular coefficient, as in (6.1).

Lemma 6.1. *Let $\alpha > 0$ and*

$$K\varphi = \frac{1}{x^\alpha}\int_0^x \frac{\varphi(t)dt}{(x-t)^{1-\alpha}}, \quad 0 < x < d. \tag{6.2}$$

The operator K is bounded in $L_p(0,d)$, if $1 < p < \infty$, and from $L_1(0,d_0)$ into $L_1((0,d_0),b)$, $b = \left(ln\ \frac{\gamma}{x}\right)^{-\lambda}$, where $0 < d_0 < d$, $\gamma > d_0$ and $\lambda > 1$.

Proof. In case $p \neq 1$, the statement of lemma is well known, being a particular case of Hardy-Littlewood theorem [27], on boundedness of integral operators with kernels homogeneous of degree -1:

$$\|K\varphi\|_p \leq B\left(\alpha, 1-\frac{1}{p}\right)\|\varphi\|_p. \tag{6.3}$$

For $p = 1$ the statement of lemma may be verified directly.

Theorem 6.2. *Let $0 < \alpha < 1$ and $f(x) \in C^+[0,d)$. If equation (6.1) is solvable in $L^+_{loc}(0,d)$, then its solution $\varphi(x)$ belongs to $C^+[0,d)$*

Proof. The proof uses the idea of the proof of Theorem 4.3.

1 step. First of all we show that the statement (4.11) is valid. By Young theorem, $t^{\alpha-1} * \varphi \in L_s(0,d_0)$ for an arbitrary $d_0 \in (0,d)$, where $1 < s < \frac{1}{1-\alpha}$. Since $\alpha + \frac{1}{s} > 1$ and $0 < \alpha < 1$ we can find s such that

$$\alpha + \frac{1}{s} < m. \tag{6.4}$$

¿From equation (6.1), $x^\alpha\varphi^m(x) \in L_s(0,d_0)$. Moreover, for the functions $\varphi(x)$ itself, we have $\varphi(x) \in L_p(0,1)$ for all

$$1 < p < \frac{m}{\alpha+\frac{1}{s}}\ . \tag{6.5}$$

Indeed, by Holder inequality with $\nu = \frac{ms}{p} > 1 + \alpha s > 1$, we have

$$\int_0^{d_0}\varphi^p(x)dx = \int_0^{d_0}\varphi^p(x)x^{\frac{\alpha s}{\nu}}x^{-\frac{\alpha s}{\nu}}dx \leq \left(\int_0^{d_0}(x^\alpha\varphi^m(x))^s\,dx\right)^{\frac{1}{\nu}}\left(\int_0^{d_0}x^{-\frac{\alpha s\nu'}{\nu}}\right)^{\frac{1}{\nu'}} < \infty.$$

Thus, we obtain (4.11) for the values of p in the interval (6.5). Now we show that (4.11) takes place for all $p > 1$. By Lemma 6.1, the integral operator on the right hand side of (6.1) preserves $L_p(0,d_0), p > 1$, so from equation (6.1), we have that $\varphi^m(x) \in L_p(0,d_0)$ or $\varphi(x) \in L_{pm}(0,d_0)$. Repeating the same arguments as in Theorem 4.3, we obtain (4.11).

2 step. We show that the function $\varphi(x)$ is in reality bounded on $[0,d_0]$. Since it is in $L_p(0,d_0)$ for all $p > 1$, the integral in the right-hand side (6.1) is a bounded continuous function. Therefore, to prove boundedness of $\varphi(x)$, we have only to show that $\varphi(x)$ is bounded at a neighborhood of the origin, say, on the interval $[0,\delta], \delta = min(1,d_0)$.

We observe that in the case of the interval $(0,\delta)$, the L_q-norm is an increasing function in q. Therefore, there exists finite or infinite limit $\lim\limits_{q\to\infty}\|\varphi\|_q \leq \infty$. It is known, see [28],p.14, that if this limit is finite, then $\varphi(x) \in L_\infty(0,\delta)$ and

$$\|\varphi\|_\infty = \lim_{q\to\infty}\|\varphi\|_q. \tag{6.9}$$

Therefore, it suffices to prove that the sequence $\|\varphi\|_q$ is bounded for all large values of q.

3 step. Taking L_q-norm in (6.1) with an arbitrary $q > 1$ and using (6.3),we have $\|\varphi\|_{qm}^m \le aB(\alpha, 1-\frac{1}{q})\|\varphi\|_q + \|f\|_\infty$. We notice that $B(\alpha, 1-\frac{1}{q}) \le B\left(\alpha, \frac{1}{2}\right)$ for all $q \ge 2$. Since we may take $q \ge 2$ by (4.11), we have

$$\|\varphi\|_q^m \le aB\left(\alpha, \frac{1}{2}\right)\|\varphi\|_q + \|f\|_\infty, \tag{6.8}$$

where we also have taken into account that $\|\varphi\|_q \le \|\varphi\|_{qm}$ by the monotonicity of the norm. Then $\|\varphi\|_q$ is uniformly bounded with respect to q by Lemma 4.1.

4 step. Since $\varphi(x) \in L_\infty^+(0,\delta)$ we have that $t^{\alpha-1} * \varphi \in C^+[0,\delta]$ and then from equation (6.1) follows that $\varphi(x) \in C^+(0,\delta]$.

5 step. It remains to prove that $\varphi(x)$ is continuous at the origin. By $\varphi_\pm$ we denote upper and lower limits of $\varphi(x)$:

$$\varphi_+ = \lim_{x\to 0} \sup_{0\le t\le x} \varphi(t), \quad \varphi_- = \lim_{x\to 0} \inf_{0\le t\le x} \varphi(t)$$

which evidently exist. Passing to $\sup\limits_{0\le t\le x}$ on the both sides of (6.1) we easily obtain:

$$\sup_{0\le t\le x} \varphi^m(t) \le \frac{a}{\alpha} \sup_{0\le t\le x} \varphi(t) + \sup_{0\le t\le x} f(t).$$

Hence

$$\varphi_+^m \le \frac{a}{\alpha}\varphi_+ + f(0).$$

Then, by Lemma 4.1.

$$\varphi_+ \le \xi_0, \tag{6.11}$$

where ξ_0 is the solution of the equation $\alpha\xi^m - a\xi - \alpha f(0) = 0$, see Lemma 4.1.

For the lower limits we similarly obtain

$$\inf_{0<t<x} \varphi^m(t) \ge \frac{a}{\alpha} \inf_{0\le t\le x} \varphi(t) + \inf_{0\le t\le x} f(t) \qquad (6.12),$$

whence

$$\varphi_-^m \ge \frac{a}{\alpha}\varphi_- + f(0)$$

which yields that

$$\varphi_- \ge \xi_0, \tag{6.13}$$

with the same ξ_0 as in (6.11). Comparing (6.11) and (6.13) we see that necessarily $\varphi_+ = \varphi_-$, so that $\varphi \in C^+[0,1]$ and $\varphi(0) = \xi_0$. The theorem is proved.

Remark 6.1. *Simple examples show that the statement of Theorem 6.2 is not valid, if $f(x)$ is only bounded, but not continuous function. For example, for the bounded function $f(x) = 0$ if $0 \le x \le d_0$ and $f(x) = 1 - \frac{a}{\alpha}\frac{(x-d_0)^\alpha}{x^\alpha}$ if $d_0 < x \le d$, $d_0 > 0$ we have the discontinuous bounded solution $\varphi(x) = 0$ if $0 \le x \le d_0$ and $\varphi(x) = 1$ if $d_0 < x \le d$.*

Corollary 6.1. *If in the conditions of Theorem 6.1 $f(x)$ is bounded: $f(x) \in L^+_\infty(0,d)$ then any solution which is apriori in $L^+_{loc}(0,d)$ in fact is in $f(x) \in L^+_\infty(0,d)$.*

Finally, we observe that the equation (6.1) in case $m < 1$ may have non-unique solution. Thus, the equation

$$\sqrt{\varphi(x)} = \frac{a}{x^\alpha}\int_0^x \frac{\varphi(t)}{(x-t)^{1-\alpha}}dt + f, \quad 0 < x < d, \tag{6.11}$$

with f is a constant, has a constant solution if c satisfy the equation $ac - \alpha\sqrt{c} + \alpha f = 0$ which has two positive solutions if $4af < \alpha$. In the linear case (m=1) the equation (6.1) is an example of linear integral equations with kernels homogeneous of degree -1, the theory of which is well developed, see [29] and also the recent survey [30]. The solvability of the equation (6.1) in C[0,d) and the number of its solutions, were in particular investigated in detail in [29].

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