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On Local Summability of Riesz Potentials in the case $\Re\alpha > 0$

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1. Introduction

It is well known that the Riesz potential operator

$$I^\alpha \varphi \ = \ \int_{R^n} k_\alpha(x-y)\varphi(y)\ dy$$

with

$$k_\alpha(x) \ = \ \frac{1}{\gamma_n(\alpha)} \begin{cases} |x|^{\alpha-n}\ , & \alpha - n \neq 0, 2, 4, 6, \cdots \\ |x|^{\alpha-n} ln\ \frac{1}{|x|}\ , & \alpha - n = 0, 2, 4, 6, \cdots \end{cases}$$

(see the value of the normalizing constant $\frac{1}{\gamma_n(\alpha)}$, for example, in [1]), is well defined for $\varphi(y) \in L_p(R^n),\ 1 \leq p < \infty$, in case of $0 < \Re\alpha < \frac{n}{p}$. It has also a sense in the case when $p = \frac{n}{\alpha},\ \alpha \in R^1$, and it is known that in this case $I^\alpha\varphi \in$ **BMO**. In the general case $\Re\alpha \geq \frac{n}{p}$, there exists a known way to define $f = I^\alpha\varphi$ as a distribution over the Lizorkin test function space Φ :

$$(I^\alpha\varphi, \omega) \ = \ (\varphi, I^\alpha\omega)\,, \quad \omega \in \Phi,\ \varphi \in L_p\ , \tag{1}$$

based on the fact that I^α preserves the space Φ (see details in [2], Subsections 25.2 and 26.7 as well as references there). In this case , it was known that, although $f = I^\alpha\varphi$ of $\varphi \in L_p$ is, generally speaking, a distribution, nevertheless, f is quasisingular in the sense that finite differences $\Delta_h^\ell(I^\alpha\varphi)$ of $I^\alpha\varphi$ prove to be usual functions ($\in L_p$) for $\ell > \Re\alpha$, the differences of distributions being defined in the standard way.

The goal of this note is to prove that, in fact, any distribution $I^\alpha\varphi,\ \varphi \in L_p,\ 1 \leq p < \infty, 0 < \alpha < \infty$, is a regular distribution and even belongs to $L_p^{loc}(R^n)$. This question is close in a sense to the problem of behaviour at infinity of functions with a given L_p-behaviour of their derivatives $D^j f, |j| = \alpha = 1, 2, 3, \cdots$, which was investigated by P.I.Lizorkin [1]. In case of non-integer α, in [1] there were used the Strichartz fractional differentiation constructions.

2. Statement of the main result

Definition. By $I^\alpha(L_p),\ 1 \leq p < \infty,\ Re\alpha > 0$, we denote the space of distributions $f \in \Phi'$ represented as

$$f = I^\alpha\varphi, \quad \varphi \in L_p(R^n),$$

in the sense of (1).

By $L_p^{loc}(R^n)$ we denote the space of functions which are in L_p on any finite ball.

Theorem. $I^\alpha(L_p) \subset L_p^{loc}(R^n),\ 1 \leq p < \infty,\ \Re\alpha > 0.$

Remark. The connection between the Lizorkin and Schwartz spaces Φ' and S' is given by the factor-space relation

$$\Phi' \ = \ S'/\mathcal{P} \tag{2}$$

modulo the subspace $\mathcal{P}$ of all polynomials. So, the statement that some element $f \in S'/\mathcal{P}$ belongs to $L_p^{loc}(R^n)$ should be understood in the sense that every representative $f_o \in S'$ of the "class" $f \in S'/\mathcal{P}$ belongs to $L_p^{loc}(R^n)$.

3. Preliminaries.

The following Proposition is well known, see [3].

Proposition 1. *For any $f \in S'$, there exists a polynomial $P_m(x), x \in R^n$, such that*

$$f \ = \ P_m(D)g \ , \qquad D = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$$

where

$$g \ = \ g(x) \ \in \ C(R^n) \bigcap S'(R^n) \ . \tag{3}$$

Let

$$A_{N,k}f \;=\; \int_{R^n_+} a_N(y) f(x-ky)\, dy, \quad N=1,2,3,\cdots \;\;, \; k>0, \tag{4}$$

be the convolution operator, where

$$a_N(y) \;=\; e^{-y} y_+^{N-1} \;=\; e^{-y_1-\cdots-y_n}(y_1\cdots y_n)_+^{N-1} \;.$$

This convolution is well defined for distributions $f\in S'$:

$$(A_{N,k}f,\varphi) \;=\; \left(f, A^*_{N,k}\varphi\right), \quad \varphi\in S,$$

since the operator A^*_N preserves the space S invariant, which can be easily seen in Fourier transforms:

$$F(A^*_{N,k}\varphi) \;= const \left(\prod_{\nu=1}^{n} \frac{1}{kx_\nu - i}\right)^N \hat{\varphi}(x).$$

Lemma 1. *For any $f\in S'$ there exists N such that*

$$A_{N,k}f \in \; C(R^n)\bigcap S'(R^n) \tag{5}$$

for any $k>0$.

Proof. By Proposition 1 we have

$$(A_{N,k}f,\varphi) \;=\; \left(g, P_m(-D)A^*_{N,k}\varphi\right), \; \varphi\in S'. \tag{6}$$

Evidently,

$$F(P_m(-D)A^*_{N,k}\varphi) \;=\; const \sum_{0\le|j|\le N} a_j \frac{x_1^{j_1}\cdots x_n^{j_n}}{(kx_1-i)^N\cdots(kx_n-i)^N}\;\hat{\varphi}(x) \;, \tag{7}$$

where a_j are constants. Obviously,

$$\frac{x_1^{j_1}\cdots x_n^{j_n}}{(kx_1-i)^N\cdots(kx_n-i)^N} \;\in W_0(R^n)\;, \;\; \text{if} \;\; N<m \;,$$

$W_0(R^m)$ being the Wiener ring of Fourier transforms of functions in $L_1(R^n)$. So, $P_m(-D)A^*_{N,k}$ is an integral convolution operator:

$$P_m(-D)A^*_{N,k}\varphi \;=\; \int_{R^n} a(x-y)\varphi(y)\, dy \;\;=:\; A\varphi$$

in case of $N > m$. Then from (6)

$$(A_{N,k}f, \varphi) \ = \ (A^* g, \varphi)$$

where $g(x)$ satisfies the condition (3). To prove (5), it suffices to verify that the operator A preserves the subspace $C(R^n) \bigcap S'(R^n)$. To show this, we observe that, as it is easily seen from the structure of the Fourier transforms in (7), the operator A is a sum of convolutions with kernels of the form $D^j(e^y y_+^{N-1})$. Therefore, it suffices to consider the kernels $b(y) = e^y y_+^\lambda$, where λ is a multi-index with positive components. Since $g \in S'$ has slow growth at infinity, we can represent the convolution with this kernel as

$$e^{-y} y_+^\lambda * g(x) \ = \ \int_{R^n} b(y)(1 + |x-y|^\mu) g_0(x-y) \ dy \ , \qquad (8)$$

where $\mu > 0$ and

$$g_0(x) \ = \ (1 + |x|)^\mu g(x) \ \in \ \ C(R^n) \bigcap L_1(R^n) \ .$$

It is clear that

$$\sup_{|x| \le A} \sup_{y \in R_+^n} b(y)(1 + |x-y|)^\mu < \infty \qquad (9)$$

for any finite ball $|x| < A$. Then it is obvious that (8) is a locally bounded function. Since it is equal to

$$\int_{R^n} b(x-y)(1+|y|)^\mu g_0(y) \ dy \ ,$$

it is also easily checked that this function is continuous for any fixed value of x. $\square$

Let $\Delta_h^\ell f(x) \ = \ \sum_{k=0}^{\ell} (-1)^k \begin{pmatrix} \ell \\ k \end{pmatrix} f(x-kh)$. The following statement is known.

Lemma 2. *Let $f = I^\alpha \varphi, \varphi \in L_p(R^n), 1 \le p < \infty, \Re\alpha > 0$ and let $\ell > \Re\alpha$. Then*

$$\|\Delta_h^\ell f\|_p \ \le \ c|h|^{\Re\alpha} \ \|\varphi\|_p \ , \quad h \in R^n, \qquad (10)$$

where c does not depend on h and φ .

The proof of the estimate (10) may be found in [2], p.537, in case of real α and $1 < p < \frac{n}{\alpha}$ (we take this opportunity to note a misprint in [2], p.537:

in (26.98) there should stand $\Delta_h^\ell f$ instead of $\Delta_h^\alpha f$). The arguments there remain valid in case of complex α, because (10) is derived as a consequence of the representation

$$\Delta_h^\ell f(x) \;=\; |h|^\alpha \int_{R^n} k_{\ell,\alpha}(y)\varphi(x-|h|\omega_h(y))\; dy \tag{11}$$

valid for complex α as well. Here $k_{\ell,\alpha}(y) \in L_1(R^n)$ and $\omega_h(y)$ is a rotation in R^n , such that $\omega_h(e_1) \;=\; \frac{h}{|h|}$, $e_1 = (1, 0, \cdots, 0)$.

As regards the case $p \geq \frac{n}{\Re\alpha}$ (and $p = 1$), it is also derived from the above representation, because this representation is now valid in the distributional sense:

$$(\Delta_h^\ell f, \omega) = (\Delta_{\ell,\alpha})(\cdot, h)) * \varphi, \omega),$$

for all $\omega \in \Phi$ and all $f = I^\alpha\varphi,\; \varphi \in L_p(R^n), 1 \leq p < \infty$.

4. Proof of Theorem.

From the equality

$$\int_{R_+^n} e^{-h}h^{N-1}\Delta_h^\ell f(x)\; dh \;=\; \sum_{k=0}^{\ell}(-1)^k \begin{pmatrix} \ell \\ k \end{pmatrix} \int_{R_+^n} e^{-h}h^{N-1}f(x-kh)\; dh$$

we get

$$\mu f(x) \;=\; \int_{R_+^n} e^{-h}h^{N-1}\Delta_h^\ell f(x)\; dh \tag{12}$$

$$-\;\sum_{k=1}^{\ell}(-1)^k k^{n-N+1} \begin{pmatrix} \ell \\ k \end{pmatrix} (A_{N,k}f)(x)\;,$$

where $\mu \;=\; \int_{R_+^n} e^{-h}h^{N-1}\; dh \;\neq\; 0$ and $A_{N,k}$ are the operators (4). By Lemma 1 , $A_{N,k}f$ are continuous functions, if we take N sufficiently large. So, it suffices to show that the integral term in (12) is in L_p^{loc} . This function is even in $L_p(R^n)$ for $\ell > \Re\alpha$, because

$$\left\|\int_{R_+^n} e^{-h}h^{N-1}\Delta_h^\ell f(\cdot)\; dh\right\| \;\leq\; c\; \int_{R_+^n} e^{-h}|h|^{n(N-1)+\Re\alpha}\; dh \cdot \;\; \|\varphi\|_p$$

by Lemma 2, c being the constant from (10). □

References

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