Direct and inverse problems
of mathematical physics
(Newark, DE, 1997), 333–342,
Int. Soc. Anal. Appl. Comput., 5,
Kluwer Acad. Publ., Dordrecht, 2000
MR 2001i:46055

Denseness of $C_0^\infty(R^n)$ in the generalized Sobolev spaces $W^{m,p(x)}(R^n)$

by
Stefan Samko

1.Introduction

The spaces $L^{p(x)}(\Omega), \Omega \subseteq R^n$, with variable order $p(x)$ were studied recently. We refer to the pioneer work by I.I. Sharapudinov [6] and the later papers by O.Kováčik and J. Rákosník [2] and by the author [3]-[5]. In the paper [2] the Sobolev type spaces $W^{m,p(x)}(\Omega)$ were also studied. D.E.Edmunds and J. Rákosník [1] dealt with the problem of denseness of C^∞-functions in $W^{m,p(x)}(\Omega)$ and proved this denseness under some special monotonicity-type condition on $p(x)$. We prove that $C_0^\infty(R^n)$ is dense in $W^{m,p(x)}(R^n)$ without any monotonicity condition, requiring instead that $p(x)$ is somewhat better than just continuous - satisfies the Dini-Lipschitz condition. For this purpose we prove the boundedness of the convolution operators $\frac{1}{\epsilon^n}\mathcal{K}\left(\frac{x}{\epsilon}\right) * f$ in the space $L^{p(x)}$ uniform with respect to ϵ . This is the main result, the above mentioned denseness being its consequence, in fact.

In the one dimensional periodical case a similar result for the uniform boundedness in $L^{p(x)}$ of some family of operators K_ϵ, depending on ϵ, was proved by I.I.Sharapudinov [7].

2. Preliminaries

We refer to the papers [2]-[6] for basics of the spaces $L^{p(x)}$, but remind their definition and some important properties.

Let $p(x)$ be a measurable function on a domain $\Omega \subseteq R^n$ satisfying the condition $1 \leq p(x) \leq \infty$ and let

$$E_\infty \; = \; E_\infty(p) \; = \; \{x \in \Omega \; : p(x) = \infty\} \, .$$

We denote

$$P \; = \sup_{x\in\Omega\setminus E_\infty(p)} p(x) \; , \quad p_0 \; = \; \inf_{x\in\Omega} p(x).$$

where sup and inf stand for esssup and essinf, respectively. By $L^{p(x)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$I_p(f) \; := \quad \int_{\Omega\setminus E_\infty} |f(x)|^{p(x)} \, dx \; < \; \infty \quad \textit{and} \quad f(x) \in L^\infty(E_\infty).$$

Let

$$\|f\|_{(p)} \; = \; \inf\left\{\lambda > 0 \; : I_p\left(\frac{f}{\lambda}\right) \leq 1\right\} \; . \tag{1}$$

In case of $P<\infty$ the space $L^{p(x)}$ is a Banach space with respect to the norm

$$\|f\|_p \ = \ \|f\|_{(p)} \ + \ \|f\|_{L^\infty(E_\infty)} \ . \tag{2}$$

We emphasize that $\|f\|_p$ is finite for any $f(x) \in L^{p(x)}(\Omega)$ in the case $P=\infty$ as well, but $L^{p(x)}(\Omega)$ is not a linear space and $\|f\|_p$ is not a norm in this case.

We note the following properties of the space $L^{p(x)}(\Omega)$:

a) *the Hölder inequality* ([6],[2],[3]) :

$$\int_\Omega |f(x)\varphi(x)| \ dx \ \leq \ k\|f\|_p\|\varphi\|_q \tag{3},$$

where $1 \leq p(x) \leq \infty, \ \frac{1}{p(x)} \ + \ \frac{1}{q(x)} \ \equiv \ 1, \ k = \sup_{x\in\Omega}\frac{1}{p(x)} + \sup_{x\in\Omega}\frac{1}{q(x)};$

b) *inequalities between $I_p(f)$ and $\|f\|_{(p)}$* ([6],[2],[3]) :

$$\|f\|_{(p)}^P \ \leq \ I_p(f) \ \leq \ \|f\|_{(p)}^{p_0} \quad , \quad if \quad \|f\|_{(p)} \leq 1 \ , \tag{4}$$

$$\|f\|_{(p)}^{p_0} \ \leq \ I_p(f) \ \leq \ \|f\|_{(p)}^{P} \quad , \quad if \quad \|f\|_{(p)} \geq 1 \ , \tag{5}$$

the left-hand side inequality in (4) and the right-hand side one in (5) being trivial in the case $P=\infty$;

c) *estimates for the norm of the characteristic function of a set* ([3]) :

$$|E|^{\frac{1}{P}} \leq \|\chi_E\|_{(p)} \leq |E|^{\frac{1}{p_0}} \quad , \quad if \quad |E| \leq 1 \ , \quad E \subseteq \Omega\backslash E_\infty(p), \tag{6}$$

the signs of the inequalities being opposite if $|E| \geq 1$; here $|E|$ is the Lebesgue measure of E ; as in (4)-(5), the corresponding inequalities are trivial in the case $P=\infty$;

d) *the embedding theorem* ([3]) : let $1 \leq r(x) \leq p(x) \leq P < \infty$ for $x \in \Omega$ and $|\Omega| < \infty$. Then $L^{p(x)} \subseteq L^{r(x)}$ and

$$\|f\|_r \leq (a_2 + (1-a_1)|\Omega|)\|f\|_p \tag{7}$$

where $a_1 = \inf_\Omega \frac{r(x)}{p(x)}, a_2 = \sup_\Omega \frac{r(x)}{p(x)}$, see also [2] for this imbedding without the restriction $p(x) \leq P < \infty$, but with worse constants $a_2 = 1$ and $1 - a_1 = 1$.

e) *denseness of step functions* ([3]): functions of the form $\sum_{k=1}^m c_k\chi_{\Omega_k}, \Omega_k \subset \Omega, |\Omega_k| < \infty$, with constant c_k, form a dense set in $L^{p(x)}(\Omega)$.

As in [4]-[5], we use the weak Lipschits condition (Dini-Lipschits condition):

$$|p(x) \ - \ p(y)| \leq \frac{A}{\log\frac{1}{|x-y|}} \quad , |x-y| \leq \frac{1}{2} \ . \tag{8}$$

Everywhere below we assume that $P<\infty$.

3. Statements of the main results

Let $\mathcal{K}(x)$ be a measurable function with support in the ball $B_R = B(0,R)$ of a radius $R<\infty$, and let

$$\mathcal{K}_\epsilon(x) \ = \ \frac{1}{\epsilon^n}\mathcal{K}\left(\frac{x}{\epsilon}\right) \ .$$

We consider the family of operators

$$K_\epsilon f \;=\; \int_\Omega \mathcal{K}_\epsilon(x-y)f(y)dy \;, \tag{9}$$

Ω being a bounded domain in R^n .

For the given domain Ω we define the larger domain

$$\Omega_R = \{x : dist(x,\Omega) \le R\} \supseteq \Omega \;.$$

Let $p(x)$ be a function defined in Ω_R such that

$$1 \le p(x) \le P < \infty \;, \quad x \in \Omega_R \;. \tag{10}$$

Let also $\frac{1}{p(x)} \;+\; \frac{1}{q(x)} \;\equiv\; 1$ and

$$Q \;=\begin{cases} \sup_{x\in\Omega_R} q(x) = \frac{p_0}{p_0-1} \;, & \text{if } |E_1(p)| = 0 \\ \infty \;, & \text{if } |E_1(p)| > 0 \end{cases} \tag{11}$$

where $E_1(p) = \{x \in \Omega_R : p(x) = 1\}$.

Theorem 1. *Let $\mathcal{K}(x) \in L^Q(B_R)$ and let $p(x)$ satisfy (10) and (8) for all x and $y \in \Omega_R$. Then the operators K_ϵ are uniformly bounded from $L^{p(x)}(\Omega)$ into $L^{p(x)}(\Omega_R)$:*

$$\|K_\epsilon f\|_{L^{p(x)}(\Omega_R)} \le c\|f\|_{L^{p(x)}(\Omega)} \tag{12}$$

where c does not depend on ϵ.

Theorem 2 . *Let $p(x)$ and $\mathcal{K}(x)$ satisfy the assumptions of Theorem 1 and*

$$\int_{B_R} \mathcal{K}(y)dy \;=\; 1 \;. \tag{13}$$

Then (9) is an identity approximation in $L^{p(x)}(\Omega)$:

$$\lim_{\epsilon\to 0} \|K_\epsilon f \;-\; f\|_{L^{p(x)}(\Omega_R)} \;=\; 0 \;, \quad f(x) \in L^{p(x}(\Omega). \tag{14}$$

Let

$$f_\epsilon(x) \;=\; \frac{1}{\epsilon^n|B(0,1)|}\int_{y\in\Omega, |y-x|<\epsilon} f(y)dy \tag{15}$$

be the Steklov mean of the function $f(y)$.

Corollary 1. *Under the assumptions of Theorem 1 on $p(x)$,*

$$\lim_{\epsilon\to 0} \|f_\epsilon \;-\; f\|_{L^{p(x)}(\Omega)} \;=\; 0 \;. \tag{16}$$

Remark 1. The statement (16) is an analogue of mean continuity property for $L^{p(x)}$-spaces, but with respect to the averaged "shift" operator (15). In the standard form, the mean continuity property $\lim_{h\to 0} \|f(x+h) - f(x)\|_p \;=\; 0$, generally speaking, is not valid for variable exponents $p(x)$ and, moreover, there exist functions $p(x)$ and $f(x) \in L^{p(x)}$ such that $f(x+h_k) \notin L^{p(x)}$ for some $h_k \to 0$, see [2], Example 2.9 and Theorem 2.10.

Corollary 2. *Let $1 \leq p(x) \leq P < \infty, x \in R^n$, and $p(x)$ satisfy the condition (8) in any ball in R^n(where A may depend on the ball) . Then C_0^∞ is dense in $L^{p(x)}(R^n)$.*

Remark 2. As it was shown in [2], $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega), 1 \leq p(x) \leq P < \infty$, without requiring that $p(x)$ satisfies the condition (8).

Let $W^{m,p(x)} = W^{m,p(x)}(R^n)$ be the Sobolev type space of functions $f(x) \in L^{p(x)}(R^n)$ which have all the distributional derivatives $D^j f(x) \in L^{p(x)}(R^n), 0 \leq |j| \leq m$, and let

$$\|f\|_{W^{m,p(x)}} = \sum_{|j|\leq m} \|D^j f\|_p \ .$$

Theorem 3. *Let $p(x)$ satisfy the assumptions of Theorem 3. Then $C_0^\infty(R^n)$ is dense in $W^{m,p(x)}(R^n)$.*

4. Proof of Theorem 1.

We assume that

$$\|f\|_p \leq 1 \ . \tag{17}$$

By (4)-(5) it suffices to show that

$$I_p\left(K_\epsilon f\right) = \int_{\Omega_R} |K_\epsilon f(x)|^{p(x)} dx \leq c \tag{18}$$

with $c > 0$ not depending on ϵ . By the Hölder inequality (3) it is easy to show that $|K_\epsilon f(x)| \leq c$ for all $x \in \Omega_R$ and $\epsilon \geq \epsilon^o (c = c(\epsilon^o)$ in this case). Therefore, it suffices to prove (18) for $0 < \epsilon \leq \epsilon^o$ under some choice of ϵ^o.

Let

$$\Omega_R = \cup_{k=1}^N \omega_R^k$$

be any partition of Ω_R into small parts ω_R^k comparable with the given ϵ :

$$diam \ \omega_R^k \leq \epsilon \ , k = 1, 2, \cdots, N \ ; \ N = N(\epsilon).$$

We represent the integral in (18) as

$$I_p\left(K_\epsilon f\right) = \sum_{k=1}^N \int_{\omega_R^k} \left| \int_\Omega \mathcal{K}_\epsilon(x-y) f(y)\, dy \right|^{p(x)-p_k+p_k} dx \tag{19}$$

with

$$p_k = \inf_{x\in\Omega_R^k} p(x) \leq \inf_{x\in\omega_R^k} p(x) \tag{20}$$

where some larger portions $\Omega_R^k \supset \omega_R^k$ will be chosen later comparable with ϵ :

$$diam \ \Omega_R^k \leq m\epsilon \ , \qquad m > 1 \ . \tag{21}$$

We shall prove the uniform estimate

$$A_k(x,\epsilon) \ := \left| \int_\Omega \mathcal{K}_\epsilon(x-y) f(y)\, dy \right|^{p(x)-p_k} \leq c \ , \quad x \in \omega_R^k \tag{22}$$

where $c > 0$ does not depend on $x \in \omega_R^k$, k and $\epsilon \in (0, \epsilon^o)$ with some $\epsilon^o > 0$. To this end, we first obtain the estimate

$$A_k(x, \epsilon) \le c_1 \ \epsilon^{-n[p(x)-p_k]} \ , \quad x \in \Omega_R. \tag{23}$$

To get (23), we differ the cases $Q = \infty$ and $Q < \infty$.

Let $Q = \infty$. We have

$$A_k(x, \epsilon) \le \left(\frac{M}{\epsilon^n} \int_\Omega \chi_{B(0,\epsilon R)}(y)|f(y)|dy \right)^{p(x)-p_k}$$

where $M = \sup_{B_R} |\mathcal{K}(x)|$. By the Hölder inequality (3) and the assumption (17) we obtain

$$A_k(x, \epsilon) \le \left(\frac{Mk}{\epsilon^n} \|\chi_{B(0,\epsilon R)}\|_q \right)^{p(x)-p_k} \ . \tag{24}$$

According to (2) we have

$$\|\chi_{B(0,\epsilon R)}\|_q \ = \ \sup_{E_\infty(q)} \chi_{B(0,\epsilon R)}(x) \ + \ \|\chi_{B(0,\epsilon R)}\|_{(q)} = 1 + \|\chi_{B(0,\epsilon R)}\|_{(q)} \ .$$

In view of (6) we get

$$\|\chi_{B(0,\epsilon R)}\|_q \le 1 + (\epsilon^n |B(0, R)|)^{\frac{1}{q_0}} \le 2$$

under the asumption that

$$0 < \epsilon \le |B(0, R)|^{-\frac{1}{n}} \ := \epsilon_1^o \ . \tag{25}$$

Then (24) provides the estimate (23) with $c_1 = (2kM)^{P-p_0}$ if $2kM \ge 1$ and $c_1 = 1$ otherwise.

Let $Q < \infty$. The estimate (23) is obtained in a similar way. Indeed, applying the Hölder inequality (3) again, we arrive at

$$A_k(x, \epsilon) \le (k\|\mathcal{K}_\epsilon(x - y)\|_q)^{p(x)-p_k} \ .$$

By (4)-(5) we have

$$\|\mathcal{K}_\epsilon(x - y)\|_{(q)} = \frac{1}{\epsilon^n} \|\mathcal{K}\left(\frac{x - y}{\epsilon}\right)\|_{(q)} \le \frac{1}{\epsilon^n} \left(\int_{\Omega \setminus E_\infty(q)} \left|\mathcal{K}\left(\frac{x - y}{\epsilon}\right)\right|^{q(y)} dy \right)^\theta$$

where $\theta = \frac{1}{Q}$ or $\theta = \frac{1}{q_o}$ depending on the fact whether the last integral in the parentheses is less or greater than 1, respectively. Hence,

$$\|\mathcal{K}_\epsilon(x - y)\|_{(q)} \le \frac{1}{\epsilon^n} \left(\int_{|y|<R, x-\epsilon y \in \Omega \setminus E_\infty(q)} |\mathcal{K}(y)|^{q(x-\epsilon y)} \ dy \right)^\theta$$

$$\le \frac{1}{\epsilon^n} \left[|B_R| \ + \ \int_{|y|<R, |\mathcal{K}(y)| \ge 1} |\mathcal{K}(y)|^Q \ dy \right]^\theta \le \frac{1}{\epsilon^n} \left[|B_R| \ + \ \|\mathcal{K}\|_Q^Q \right]^\theta \le c_2 \epsilon^{-n} \tag{27}$$

where $c_2 = \max\{c_3^{\frac{1}{Q}}, c_3^{\frac{1}{q_o}}\}, c_3 = |B_R| + \|\mathcal{K}\|_Q^Q$.

Therefore, from (26) and (27) we obtain (23) in the case $Q < \infty$ as well, with $c_1 = (c_2 k)^{P-p_o}$ if $c_2 > 1$ and $c_1 = 1$ otherwise.

The estimate (23) having been proved, we observe now that by (8)

$$p(x) - p_k = |p(x) - p(\xi_k)| \leq \frac{A}{\log \frac{1}{|x-\xi_k|}}$$

where $x \in \omega_R^k,\ \xi_k \in \Omega_R^k$. Evidently,

$$|x - \xi_k| \leq diam\Omega_R^k \leq m\epsilon$$

by (21). Therefore,

$$p(x) - p_k \leq \frac{A}{\log \frac{1}{m\epsilon}} \tag{28}$$

under the assumption that

$$0 < \epsilon \leq \frac{1}{2m} \ =: \ \epsilon_2^o \ . \tag{29}$$

Then from (23) and (28)

$$A_k(x, \epsilon) \ \leq \ c_1 \epsilon^{-\frac{A}{\log \frac{1}{m\epsilon}}} \ , \ x \in \omega_R^k \ , \tag{30}$$

c_1 not depending on x and being given above. Then from (30)

$$A_k(x, \epsilon) \ \leq \ c_4 \ := \ c_1 e^{2A}$$

for $x \in \omega_R^k$ and

$$0 < \epsilon \leq \epsilon_3^0 \ := \frac{1}{m^2}. \tag{31}$$

Therefore, we have the uniform estimate (22) with $c = c_1 e^{2A}$ and $0 < \epsilon \leq \epsilon^o$, $\epsilon^o = \min_{1 \leq k \leq 3} \epsilon_k^0$, ϵ_k^o being given by (25), (29) and (31).

Using the estimate (22) we obtain from (19)

$$I_p\left(K_\epsilon f\right) \ \leq \ c \sum_{k=1}^{N} \int_{\omega_R^k} \left| \int_\Omega \mathcal{K}_\epsilon(x-y) f(y) dy \right|^{p_k} dx \ .$$

Here p_k are constants so that we may apply the usual Minkowsky inequality for integrals and obtain

$$I_p\left(K_\epsilon f\right) \ \leq \ c \sum_{k=1}^{N} \left\{ \int_{|y|<\epsilon R} |\mathcal{K}_\epsilon(y)|\, dy \left(\int_{\omega_R^k} |f(x-y)|^{p_k} dx \right)^{\frac{1}{p_k}} \right\}^{p_k}$$

$$= c \sum_{k=1}^{N} \left\{ \int_{|y|<R} |\mathcal{K}(y)|\, dy \left(\int_{x+\epsilon y \in \omega_R^k} |f(x)|^{p_k} dx \right)^{\frac{1}{p_k}} \right\}^{p_k} \ . \tag{32}$$

Obviously, the domain of integration in x in the last integral is embedded into the domain

$$\bigcup_{y\in B_{\epsilon R}} \{x : x+y \in \omega_R^k\} \tag{33}$$

which already does not depend on y . Now, we choose the sets Ω_R^k in (20), which were not determined until now, as the sets (33). Then, evidently, $\Omega_R^k \supset \omega_R^k$ and it is easily seen that

$$diam\ \Omega_R^k \le (1+2R)\epsilon \tag{34}$$

so that the requirement (21) is satisfied with $m = 1+2R$.

From (32) we have

$$I_p\left(K_\epsilon f\right) \ \le\ c\sum_{k=1}^{N} \left\{\int_{|y|<R} |\mathcal{K}(y)|\,dy\right\}^{p_k} \int_{\Omega_R^k} |f(x)|^{p_k}\,dx$$

$$\le c\left\{\int_{|y|<R} |\mathcal{K}(y)|\,dy\right\}^{\theta} \sum_{k=1}^{N} \int_{\Omega_R^k\cap\Omega} |f(x)|^{p_k}\,dx$$

where $\theta = P$ if $\int_{|y|<R} |\mathcal{K}(y)|dy \le 1$ and $\theta = p_o$ otherwise. In view of (34), the covering $\{\omega_k = \Omega_R^k \cap \Omega\}_{k=1}^N$ has a finite multiplicity (that is, each point $x \in \Omega$ belongs simultaneously not more than to a finite number n_o of the sets ω_k, $n_o \le 1 + (1+2R)^n$ in this case). Therefore,

$$I_p\left(K_\epsilon f\right) \ \le\ c_5 \int_\Omega |f(x)|^{\tilde{p}(x)}\,dx \tag{35}$$

where

$$\tilde{p}(x) \ = \ \max_j p_j$$

the maximum being taken with respect to all the sets ω_j containing x . Evidently, $\tilde{p}(x) \le p(x)$ for $x \in \Omega$. Then from (35) and (4)-(5) we obtain the estimate

$$I_p\left(K_\epsilon f\right) \ \le\ c_5\|f\|_{\tilde{p}}^{\theta_1} \ , \quad \theta_1 < P,$$

with $\theta_1 = \inf \tilde{p}(x)$ if $\|f\|_{\tilde{p}} \le 1$ and $\theta_1 = \sup \tilde{p}(x)$ otherwise. Applying the imbedding theorem (7), we arrive at the final estimate

$$I_p\left(K_\epsilon f\right) \ \le\ c_6\|f\|_p^{\theta_1} \le c_6 \ .$$

5. Proof of Theorem 2.

To prove (14), we use Theorem 1, which provides the uniform boundedness of the operators K_ϵ from $L^{p(x)}(\Omega)$ into $L^{p(x)}(\Omega_R)$. Then, by the Banach-Steinhaus theorem it suffices to verify that (14) holds for some dense set in $L^{p(x)}(\Omega)$, for example, for step functions, according to property e) of the spaces $L^{p(x)}(\Omega)$. So, it is sufficient to prove (14) for the characteristic function $\chi_E(x)$ of any bounded measurable set $E \subset \Omega$. We have

$$K_\epsilon(\chi_E) \ - \ \chi_E \ = \ \int_{B_R} \mathcal{K}(y)\left[\chi_E(x-\epsilon y) - \chi_E(x)\right]dy$$

by (13). Hence

$$\|K_\epsilon(\chi_E) \ - \ \chi_E\|_P \ \leq \ \int_{B_R} |\mathcal{K}(y)| \, \|\chi_E(\cdot - \epsilon y) - \chi_E(x)\|_P \, dy \to \ 0$$

as $\epsilon \to 0$ by the Lebesgue dominated convergence theorem and the P-mean continuity of functions in L^P with a constant P ($P = \sup_{x \in \Omega_R} p(x)$ in this case). Then, by (7), also

$$\|K_\epsilon(\chi_E) \ - \ \chi_E\|_p \to 0$$

with $p = p(x) \leq P < \infty$.□

6. Proof of Corollaries

To obtain Corollary 1 from Theorem 1, it suffices to choose $\mathcal{K}(y) = \frac{1}{|B(0,1)|}\chi_{B(0,1)}(y)$.

Proof of Corollary 2. Let $\chi_N(x) = \chi_{B(0,N)}(x)$. Then the functions $f^N(x) = \chi_N(x)f(x)$ have compact support and approximate $f(x) \in L^{p(x)}(R^n)$:

$$\|f - f^N\| \leq I_p^{\frac{1}{P}}(f - f^N) = \left(\int_{|x|>N} |f(x)|^{p(x)} dx \right)^{\frac{1}{P}} \to 0$$

as $N \to \infty$.

Therefore, we may consider $f(x)$ with a compact support in the ball B_N from the very beginning. To approximate $f(x)$ by C_0^∞, we use the identity approximation

$$f_\epsilon(x) \ = \int_{R^n} \mathcal{K}_\epsilon(x - t) f(t) \, dt \ = \int_{|y|<1} \mathcal{K}(y) f(x - \epsilon y) \, dy \tag{36}$$

where $\mathcal{K}_\epsilon(x) = \frac{1}{\epsilon^n}\mathcal{K}\left(\frac{x}{\epsilon}\right)$ and $\mathcal{K}(y) \in C_0^\infty(R^n)$ with support in the ball B_1 and such that

$$\int_{|y|<1} \mathcal{K}(y) \, dy = 1 \ .$$

Then, evidently, $f_\epsilon(x) \in C_0^\infty(R^n)$ and has compact support because $f_\epsilon(x) \equiv 0$ if $|x| > N + \epsilon$. Therefore, for $\epsilon < 1$,

$$\|f_\epsilon - f\|_{L^{p(x)}(R^n)} \ = \ \|K_\epsilon f - f\|_{L^{p(x)}(B_{N+1})} \to 0$$

as $\epsilon \to 0$.

Proof of Theorem 3.

The proof follows from Theorem 2 and Corollary 2 in two steps.

1^o. Let $f(x) \in W^{m,p(x)}(R^n)$ and let $\mu(r), 0 \leq r \leq \infty$, be a smooth step-function: $\mu(r) \equiv 1$ for $0 \leq r \leq 1, \mu(r) \equiv 0$ for $r \geq 2, \mu(r) \in C_0^\infty(R_+^1)$ and $0 \leq \mu(r) \leq 1$. Then

$$f^N(x) \ = \ \mu\left(\frac{|x|}{N}\right) f(x) \in W^{m,p(x)}(R^n) \tag{37}$$

for every $N \in R_+^1$ and has compact support in B_{2N}.

The functions (37) approximate $f(x)$ in $W^{m,p(x)}(R^n)$. Indeed, denoting $\nu_N(x) = 1 - \mu\left(\frac{|x|}{N}\right)$, so that $\nu_N(x) \equiv 0$ for $|x| < N$, and using the Leibnitz formula for differentiation, we have

$$\begin{aligned} \|f - f^N\|_{W^{m,p(x)}} &= \sum_{|j|\le m} \|D^j(\nu_N f)\|_p \le \sum_{|j|\le m}\sum_{0\le k\le j} c_k \|D^k(\nu_N)D^{j-k}f\|_p \\ &\le \sum_{|j|\le m} \|\nu_N D^j f\|_p \ + \ c\sum_{|j|\le m}\sum_{0<k\le j} \|D^k(\nu_N)D^{j-k}f\|_p \\ &\le \sum_{|j|\le m} \|\nu_N D^j f\|_p + c\sum_{|j|\le m}\sum_{0<k\le j} \frac{1}{N^{|k|}}\|D^{j-k}f\|_p \ \to 0 \end{aligned} \tag{38}$$

as $N \to 0$.

2. By the step 1^o we may consider $f(x) \in W^{m,p(x)}$with compact support. Then we take $\mathcal{K}(y) \in C_0^\infty(R^n)$ with support in the ball B_1 and such that $\int_{|y|<1} \mathcal{K}(y)dy = 1$ and arrange the approximation (36). Then, evidently, $f_\epsilon \in C_0^\infty(R^n)$. Indeed, for any j we have

$$D^j f_\epsilon(x) \ = \ \frac{1}{\epsilon^{n+|j|}} \int_{|y|<1} (D^j\mathcal{K})\left(\frac{x-t}{\epsilon}\right) f(t)dt \in C^\infty(R^n)$$

and $f_\epsilon(x)$ has compact support because $f_\epsilon(x) \equiv 0$ if $|x| > 1+\lambda$, where $\lambda = \sup_{x\in supp\ f} |x|$, *supp* standing for support of $f(x)$.

We have

$$\begin{aligned} \|f_\epsilon(x) - f\|_{W^{m,p(x)}} &\le \sum_{|j|\le m} \|D^j f - K_\epsilon(D^j f)\|_{L^{p(x)}(R^n)} \\ &= \sum_{|j|\le m} \|D^j f - K_\epsilon(D^j f)\|_{L^{p(x)}(\Omega_1)} \end{aligned}$$

where $\Omega_1 = \{x : dist(x,\Omega) \le 1\}, \Omega = supp f(x)$. It suffices to apply Theorem 2.

References

1 . Edmunds,D.E. and Rákosník, J. Density of smooth functions in $W^{k,p(x)}(\Omega)$. *Proc. R. Soc. Lond.* **A** (1992)**437,** 229-236.

2 . Kováčik, O. and Rákosník, J. On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czech. Math. J.,* **41(116)(**1991), 592-618.

3 . Samko, S.G. Differentiation and integration of variable order and the spaces $L^{p(x)}$. *Proceed. of Intern. Conference "Operator Theory and Complex and Hypercomplex Analysis", 12-17 December 1994, Mexico City, Mexico, Contemp. Math.,* 1997

4 . Samko S.G. Convolution type operators in $L^{p(x)}$. Preliminary results. *Integr. Transf. Spec. Funct.,* 1997 (to appear).

5 . Samko S.G. Convolution and potential type operators in $L^{p(x)}$. *Integr. Transf. Spec. Funct.,* 1997 (to appear).

6 . Sharapudinov, I.I. On a topology of the space $L^{p(t)}([0,1])$. *Matem. Zametki* **26**(1979), no 4, 613-632.

7 . Sharapudinov, I.I. On uniform boundedness in $L^p (p = p(x))$ of some families of convolution operators (in Russian). *Matem. Zametki,* **59** (1996), no. 2, 291-302.