To appear in
Proceedings of the Conference
IWOTA-2000, Faro
Birkhäuser, 2002

On Inversion of Fractional Spherical Potentials by Spherical Hypersingular Operators

Stefan Samko

A new proof of the inversion formula for spherical Riesz type fractional potentials in the case $0 < \Re\alpha < 2$ is presented and a constructive reduction of the case $\Re\alpha > 2$ to the case $0 < \Re\alpha < 2$ is given.

1. Introduction

Let $S^{n-1} = \{x \in R^n : |x| = 1\}$ be the unit sphere in R^n. The fractional spherical potentials, named also Riesz spherical potentials, are known in the form

$$(1.1)\quad (K^\alpha\varphi)(x) = \frac{1}{\gamma_{n-1}(\alpha)} \int_{S^{n-1}} \frac{\varphi(\sigma)d\sigma}{|x-\sigma|^{n-1-\alpha}}, \quad \alpha > 0,\ \alpha \neq n-1, n+1, n+3, ...$$

where $|x| = 1$ and the normalizing constant $\gamma_{n-1}(\lambda)$ is defined by the formula

$$(1.2)\qquad \gamma_{n-1}(\alpha) = \frac{2^\alpha \pi^{\frac{n-1}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-1-\alpha}{2}\right)}\ .$$

The inversion of this potential type operator was given more than two decades ago in [PS1],[PS2] in the case $0 < \alpha < 2$ in terms of spherical hypersingular integrals. We here take up the story again and present another proof, which is more transparent and effective and also covers the case $0 < \Re\alpha < 2$. As for the case $\Re\alpha > 2,\ \alpha - n \neq 2k+1,\ k = 0, 1, 2, ...,$ we show that this case may be constructively reduced to the case $0 < \Re\alpha < 2$.

The main statements are given in Theorems 3.5 and 4.1 below.

2. Preliminaries

a) Spherical multiplier of the Riesz potential operator. We use the basics of the theory of spherical harmonics and Fourier-Laplace expansions into series of spherical harmonics. We refer, for instance, to the books [SW], [Mu], or [Sa], Section 2 of Ch.1. In particular, we use the notion of spherical Laplace-Fourier multipliers.

It is known that in the case $\alpha \neq n-1+2k, k = 0,1,2,...$, the Laplace-Fourier multiplier of the operator K^α is equal to

$$\{k_m^\alpha\}_{m=0}^\infty = \left\{ \frac{\Gamma\left(m+\frac{n-1-\alpha}{2}\right)}{\Gamma\left(m+\frac{n-1+\alpha}{2}\right)} \right\}_{m=0}^\infty . \tag{2.1}$$

b) Spherical hypersingular integrals of order $0 < \Re\alpha < 2$. The spherical hypersingular integral is introduced ([PS]) as

$$(D^\alpha f)(x) = \frac{1}{\gamma_{n-1}(-\alpha)} \int_{S^{n-1}} \frac{f(\sigma)-f(x)}{|x-\sigma|^{n-1-\alpha}}\, d\sigma, \qquad x \in S^{n-1} , \tag{2.2}$$

which converges absolutely for $0 < \Re\alpha < 1$ in the case of "nice" functions $f(x)$. It converges also for $1 \leq \Re\alpha < 2$ in the case of sufficiently "nice" functions $f(\sigma)$ provided the integral is interpreted as the limit of *truncated* hypersingular operators

$$(D_\varepsilon^\alpha f)(x) = \frac{1}{\gamma_{n-1}(-\alpha)} \int_{S_\varepsilon^{n-1}(x)} \frac{f(\sigma)-f(x)}{|x-\sigma|^{n-1-\alpha}}\, d\sigma, \qquad x \in S^{n-1}, \tag{2.3}$$

where $S_\varepsilon^{n-1}(x) = \{\sigma \in S^{n-1} : |\sigma - x| > \epsilon\}$. For completeness we prove this convergence in the lemma below.

The integral (2.3) is known ([PS1]) to be representable in the form

$$(D_\varepsilon^\alpha f)(x) = \frac{|S^{n-2}|}{\gamma_{n-1}(-\alpha)2^{(n+\alpha-1)/2}} \int_{-1}^{1-\frac{\varepsilon}{2}} \frac{M_f(x,t)-f(x)}{(1-t)^{\frac{n-1+\alpha}{2}}}(1-t^2)^{\frac{n-3}{2}}\, dt \tag{2.4}$$

where $M_f(x,t)$ are the means

$$M_f(x,t) = \frac{1}{|S^{n-2}|(1-t^2)^{\frac{n-2}{2}}} \int_{\substack{|\sigma|=1\\ \sigma\cdot x=t}} f(\sigma)\, ds_\sigma. \tag{2.5}$$

The representation (2.4) follows from (2.3) by the formula

$$\int_{S^{n-1}} f(\sigma)\varphi(x\cdot\sigma)\, d\sigma = |S^{n-2}| \int_{-1}^{1} M_f(x,t)(1-t^2)^{\frac{n-3}{2}}\varphi(t)\, dt \tag{2.6}$$

which can be named Cavaliery type principle; for its proof see [Sa2] or [Sa], Lemma 4.13.

Writing that $f(\sigma) \in C^N(S^{n-1})$, we mean that $f\left(\frac{x}{|x|}\right) \in C^N(R^n \backslash \{0\})$.

Lemma 2.1. *Let $f(\sigma) \in C^{2N}(S^{n-1}),\ N > \frac{3}{4}n$. Then the limit*

$$\lim_{\varepsilon \to 0} D_\varepsilon^\alpha f(x)$$

exists in $C(S^{n-1})$ for every α with $0 < \Re\alpha < 2$.

Proof. We expand $f(x)$ in a Fourier-Laplace series of spherical harmonics, which converges by the known properties of series of spherical harmonics. We have

$$(D_\varepsilon^\alpha f)(x) = \frac{1}{\gamma_{n-1}(-\alpha)} \sum_{m\mu}^{\infty} \int_{2(1-x\cdot\sigma)>\varepsilon^2} \frac{Y_{m\mu}(\sigma) - Y_{m\mu}(x)}{[2(1-x\cdot\sigma)]^{\frac{n-1+\alpha}{2}}}\, d\sigma\ . \tag{2.7}$$

We make use of the Funk-Hekke formula

$$\int_{S^{n-1}} f(x\cdot\sigma) Y_m(\sigma)\, d\sigma = \lambda Y_m(x), \tag{2.8}$$

where

$$\lambda = |S^{n-2}| \int_{-1}^{1} f(t) P_m(t) (1-t^2)^{\frac{n-3}{2}}\, dt \tag{2.9}$$

and $P_m(t)$ is the m-th Legendre polynomial. As is known, $P_m(t)$ can be expressed in terms of the Gegenbauer polynomials via

$$P_m(t) = \binom{m+n-3}{m}^{-1} C^{\frac{n-2}{2}}(t), \qquad n \geq 3 \tag{2.10}$$

(for more details on the Funk-Hekke formula, see, for example, [Sa], Section 2 of Ch.1). Formula (2.8) is valid for any function $f(t)$ such that $f(t)(1-t^2)^{\frac{n-3}{2}} \in L_1([-1,1])$. By means of this formula we obtain

$$(D_\varepsilon^\alpha f)(x) = \frac{1}{2^{\frac{n-1+\alpha}{2}} \gamma_{n-1}(-\alpha)} \sum_{m\mu}^{\infty} f_{m\mu} b_{m,\varepsilon}(\alpha) Y_{m\mu}(x) \tag{2.11}$$

where

$$b_{m,\varepsilon}(\alpha) = \int_{-1}^{1-\frac{\varepsilon^2}{2}} (1+t)^{\frac{n-3}{2}} (1-t)^{-1-\frac{\alpha}{2}} [P_m(t) - 1]\, dt. \tag{2.12}$$

To estimate this coefficient, we observe that

$$|P_m'(t)| \leq \frac{m(m+n-2)}{n-1}\ , \tag{2.13}$$

which follows from (2.10) and the properties

$$\frac{d}{dt}C_m^\lambda(t) = 2\lambda C_{m-1}^{\lambda+1}(t), \qquad |C_m^\lambda(t)| \le \frac{\Gamma(m+2\lambda)}{m!\Gamma(2\lambda)}$$

of Gegenbauer polynomials. From (2.13) the estimate

$$|P_m(t) - 1| \le \frac{m(m+n-2)}{n-1}(1-t)$$

follows. Hence

$$|b_{m,\epsilon}(\alpha)| \le cm^2$$

with c not depending on m and ε . Therefore, to demonstrate the uniform convergence of the series (2.11), it suffices to prove the uniform convergence of the series $\sum_{m=0}^{\infty} m^2 |f_{m\mu} Y_{m\mu}(x)|$. But this follows from known theorems on the convergence of Fourier-Laplace series of smooth functions (see [Ne], p.232) with the inequality $|Y_{m\mu}(x)| \le cm^{\frac{n-2}{2}}$ taken into account. □

c) The multiplier of the spherical hypersingular operator. From the expansion (2.11) we obtain that the operator (2.2) has the spherical Fourier multiplier λ_m (that is, $D^\alpha Y_m = \lambda_m Y_m$) given by

$$\lambda_m = \frac{|S^{n-2}|}{2^{\frac{n-1+\alpha}{2}}\gamma_{n-1}(\alpha)} b_{m,0}(\alpha) \ . \tag{2.14}$$

To calculate $b_{m,0}(\alpha)$, we note that $b_{m,0}(\alpha)$ is an analytic function of the parameter α in the half-plane $\Re\alpha < 2$. Taking $\Re\alpha < 0$, we easily calculate $b_{m,0}(\alpha)$ by means of formula 7.311.3 of [GR], which yields the formula

$$\lambda_m = \frac{\Gamma\left(m + \frac{n-1+\alpha}{2}\right)}{\Gamma\left(m + \frac{n-1-\alpha}{2}\right)} - \frac{\Gamma\left(\frac{n-1+\alpha}{2}\right)}{\Gamma\left(\frac{n-1-\alpha}{2}\right)}. \tag{2.15}$$

As a consequence of (2.1) and (2.15), the operator $(K^\alpha)^{-1}$ inverse to the spherical potential operator K^α is expected to be

$$(K^\alpha)^{-1} = c_\alpha I + D^\alpha, \qquad c_\alpha = \frac{\Gamma\left(\frac{n-1+\alpha}{2}\right)}{\Gamma\left(\frac{n-1-\alpha}{2}\right)} \ , \tag{2.16}$$

where I is the identity operator.

d) Stereographic projection. The change of variables in R^n defined by

$$\xi = s(x) = \{s_1(x), \cdots, s_n(x)\} \tag{2.17}$$

with $s_k(x) = \frac{2x_k}{1+|x|^2}$, $k = 1, \cdots, n-1$, and $s_n(x) = \frac{|x|^2-1}{|x|^2+1}$, $x = (x_1, \cdots, x_n) \in R^n$, $|x| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$, generates the mapping known as the *stereographic* *projection.* It maps the subspace $\dot{R}^{n-1}$ one-to-one onto the unit sphere $S^{n-1} \subset R^n$, $\dot{R}^{n-1}$ being the completion of R^{n-1} by a single infinite point. It is known, see [Mi],p.35-36, that the following relations hold:

$$|\xi - \sigma| = \frac{2|x-y|}{(1+|x|^2)^{\frac{1}{2}}(1+|y|^2)^{\frac{1}{2}}}, \qquad \xi = s(x), \quad \sigma = s(y), \tag{2.18}$$

$$d\sigma = \frac{2^{n-1}dy}{(1+|y|^2)^{n-1}}, \qquad \xi = s(x), \quad \sigma = s(y). \tag{2.19}$$

This immediately yields the relation

$$\int_{S^{n-1}} \frac{f(\sigma)\, d\sigma}{|\xi-\sigma|^{n-1-\alpha}} = 2^\alpha (1+|x|^2)^{\frac{n-1-\alpha}{2}} \int_{R^{n-1}} \frac{f[s(y)]\, dy}{|x-y|^{n-1-\alpha}(1+|y|^2)^{\frac{n-1-\alpha}{2}}}\ , \tag{2.20}$$

$s(y)$ being the stereographic projection of R^{n-1} onto S^{n-1}.

3. Justification of the inversion in $L_p(S^{n-1})$ and identity approximation on the sphere

We wish to show that the operator $c_\alpha + D^\alpha$ interpreted as $c_\alpha I + \lim_{\substack{\epsilon\to 0\\(L_p)}} D^\alpha_\epsilon$ is a left inverse to the operator K^α in the spaces $L_p(S^{n-1})$:

$$(c_\alpha I + \lim_{\substack{\epsilon\to 0\\(L_p)}} D^\alpha_\epsilon) K^\alpha \varphi = \varphi, \qquad \varphi \in L_p(S^{n-1})\ . \tag{3.1}$$

Direct calculations yield

$$(D^\alpha_\epsilon K^\alpha \varphi)(x) \ = \int_{S^{n-1}} \mathcal{L}_\epsilon(x\cdot\sigma)\varphi(\sigma)d\sigma \tag{3.2}$$

where the kernel $\mathcal{L}_\epsilon(x\cdot\sigma)$ is defined by

$$\mathcal{L}_\varepsilon(x\cdot\sigma) = \frac{1}{\gamma}\int_{S^{n-1}_\varepsilon(x)} \frac{|\tau-\sigma|^{1-n+\alpha} - |x-\sigma|^{1-n+\alpha}}{|\tau-x|^{n-1+\alpha}}\, d\tau \tag{3.3}$$

and $\gamma = \gamma_{n-1}(\alpha)\gamma_{n-1}(-\alpha) = -d_{n-1,1}(\alpha)\gamma_{n-1}(\alpha)$. We know beforehand that the kernel in (3.3) may depend only on the inner product $x\cdot\sigma$ because $D^\alpha_\epsilon K^\alpha$ is a composition of two spherical convolution operators.

As we shall see, the kernel $\mathcal{L}_\epsilon(x\cdot\sigma)$ involves an identity approximation kernel, so we dwell on this notion. We say that the spherical convolution operator

$$(A_\epsilon\varphi)(x) \ = \int_{S^{n-1}} \mathcal{A}_\epsilon(x\cdot\sigma)\varphi(\sigma)d\sigma \tag{3.4}$$

is a spherical identity approximation operator in $L_p(S^{n-1})$ if

$$\lim_{\substack{\epsilon\to 0\\(L_p)}} \|A_\epsilon\varphi-\varphi\|_{L_p(S^{n-1})}=0\ .$$

The following theorem ([BBP], p.210) provides sufficient conditions for a kernel $\mathcal{A}_\epsilon(x\cdot\sigma)$ to be an identity approximation kernel.

Theorem 3.1. *Let $\mathcal{A}_\epsilon(t)$ satisfy the conditions*
i) $\lim_{\epsilon\to 0}\int_{S^{n-1}}\mathcal{A}_\epsilon(x\cdot\sigma)d\sigma=1$*;*
ii) $\int_{S^{n-1}}|\mathcal{A}_\epsilon(x\cdot\sigma)|d\sigma\le M<\infty,\ 0<\epsilon<\epsilon_0$*, with M not depending on ϵ;*
iii) $\lim_{\epsilon\to 0}\int_{x\cdot\sigma<1-t_0}|\mathcal{A}_\epsilon(x\cdot\sigma)|d\sigma=0$ *for each $t_0\in(0,2)$.*
Then the operator (3.4) is an identity approximation in $L_p(S^{n-1}),\ 1\le p<\infty$ (and in $C(S^{n-1})$ as well).

The main job to be done now is to single out the identity approximation term in the kernel (3.3), which will be accomplished in Lemma 3.3 below. Lemma 3.2 is crucial for this purpose.

Let $u\in R^{n-1}$ be the vector connected with $x,\sigma\in S^{n-1}$ by the relation

$$s(u)\ =\ \omega(\sigma), \tag{3.5}$$

where s is the stereographical projection (2.17) and $\omega(\sigma)=\omega_x(\sigma)$ is any rotation of the sphere such that

$$\omega(x)=-e_n=(0,0,...,0,-1). \tag{3.6}$$

We also use the notation

$$\langle u\rangle=\sqrt{1+|u|^2}\ .$$

Lemma 3.2. *Let $x,\sigma\in S^{n-1}$. Then*

$$\int_{S^{n-1}} f(|x-\tau|,|\sigma-\tau|)\,d\tau=2^{n-1}\int_{R^{n-1}} f\left(\frac{2|v|}{\langle v\rangle},\frac{2|v|}{\langle v\rangle}\right)\frac{dv}{\langle v\rangle^{2(n-1)}} \tag{3.7}$$

under the assumption that the integrals converge. Here u is the vector defined in (3.5), so that

$$|u|=\frac{|x-\sigma|}{|x+\sigma|},\ \langle u\rangle=\frac{2}{|x+\sigma|},\ \frac{\langle u\rangle}{|u|}=\frac{2}{|x-\sigma|}\ . \tag{3.8}$$

Proof. Making the rotation change of variables $\eta=\omega(\tau)$ with the rotation (3.6), we transform the left-hand side of (3.7) to

$$\int_{S^{n-1}} f(|\eta+e_n|,|\xi-\eta|)d\eta.$$

After that, we may apply the stereographical projection (2.17) by putting

$$\eta = s(v), \quad v \in R^{n-1},$$

(so that $-\eta_n = s(0)$). Making use of the relations (2.18) and (2.19), we arrive at formula (3.7).

To check the relations (3.8), we denote $\xi = \omega(\sigma)$ and have

$$x \cdot \sigma = \omega^{-1}(e_n) \cdot \omega^{-1}(\xi) = e_n \cdot \xi = -\xi_n.$$

Since (3.5) implies $s_n(u) = \xi_n$, we obtain $\xi_n = \frac{|u|^2-1}{|u|^2+1}$. Therefore,

$$|u|^2 = \frac{1+\xi_n}{1-\xi_n} = \frac{1-x\cdot\sigma}{1+x\cdot\sigma} = \frac{|x-\sigma|^2}{|x+\sigma|^2}, \tag{3.9}$$

from which the first of the relations (3.8) follows, the two others being its consequences. □

Remark. The representation (3.7) may be rewritten in the form

$$\int_{S^{n-1}} f(|x-\tau|, |\sigma-\tau|)\, d\tau = 2^{n-1} \int_{R^{n-1}} f\left(\frac{2|y|}{\langle y\rangle}, \frac{2|re_1-y|}{\sqrt{1+r^2}\langle y\rangle}\right) \frac{dy}{\langle y\rangle^{2(n-1)}} \tag{3.10}$$

where $r = |u| = \frac{|x-\sigma|}{|x+\sigma|} = \frac{1-x\cdot\sigma}{1-x\cdot\sigma}$, which obviously reveals the dependence of this integral on the inner product $x \cdot \sigma$ only.

To get (3.10) from (3.7), it suffices to make another change of variables, namely, the rotation $v = rot\ y$ in R^{n-1} which rotates e_1 to $\frac{u}{|u|}$: $rot\ e_1 = \frac{u}{|u|}$. Then $|u-v| = ||u|e_1 - y|$, which yields (3.10).

In the following lemma on the representation of the kernel $\mathcal{L}_\epsilon(x\cdot\sigma)$ introduced in (3.2), we use the kernel

$$\mathcal{K}_{1,\alpha}(r) = \frac{1}{\gamma r^{n-1}} \int_{|y|\langle r} \left(|y-e_1|^{\alpha-n+1} - |y|^{\alpha-n+1}\right)\, dy \tag{3.11}$$

with the integration over the ball in R^{n-1}. We remark that this kernel is a direct analog of a similar kernel familiar in the theory of spatial hypersingular integrals (see [Sa1]; [SKM], formula (26.30); [Sa], formula (3.46)).

Lemma 3.3. *The kernel $\mathcal{L}_\epsilon(x\cdot\sigma)$ may be represented as*

$$\mathcal{L}_\epsilon(x\cdot\sigma) \;=\; \frac{c(\epsilon)}{|x-\sigma|^{n-1-\alpha}} \;+\; \mathcal{K}_\epsilon(x\cdot\sigma), \tag{3.12}$$

where

$$c(\varepsilon) = \frac{|S^{n-2}|}{\gamma 2^\alpha} \int_\delta^\infty \frac{1-(1+r^2)^{\frac{\alpha+1-n}{2}}}{r^{1+\alpha}}\, dr, \qquad \delta = \frac{\varepsilon}{\sqrt{4-\varepsilon^2}}, \tag{3.13}$$

and

$$\mathcal{K}_\varepsilon(x\cdot\sigma) = \frac{2^{-\alpha}}{|x+\sigma|^{n-1-\alpha}}\ \frac{1}{\delta^{n-1}}\mathcal{K}_{1,\alpha}\left(\frac{|x-\sigma|}{\delta|x+\sigma|}\right) = \frac{2^{1-n}\langle u\rangle^{n-1-\alpha}}{\delta^{n-1}}\mathcal{K}_{1,\alpha}\left(\frac{|u|}{\delta}\right), \tag{3.14}$$

u being the vector defined in (3.5).

Proof. Applying formula (3.7) to the spherical integral (3.3), we arrive at the representation

$$\mathcal{L}_\varepsilon(x\cdot\sigma) = \frac{2^{1-n}\langle u\rangle^{n-1-\alpha}}{\gamma|u|^{n-1-\alpha}}\int_{R^{n-1}_\delta}\frac{|u|^{n-1-\alpha}-\left(\frac{|u-v|}{\langle v\rangle}\right)^{n-1-\alpha}}{\langle v\rangle^{n-1+\alpha}|u-v|^{n-1-\alpha}}\,dv$$

where $R^{n-1}_\delta = \{v\in R^{n-1} : |v|>\delta\}$. Hence

$$\begin{aligned}\mathcal{L}_\varepsilon(x\cdot\sigma) &= \frac{2^{1-n}}{\gamma}\left(\frac{\langle u\rangle}{|u|}\right)^{n-1-\alpha}\left(\int_{|v|>\delta}\frac{|u|^{n-1-\alpha}-|u-v|^{n-1-\alpha}}{|v|^{n-1+\alpha}|u-v|^{n-1-\alpha}}\,dv\right.\\ &\quad\left.+\int_{|v|>\delta}\frac{1-\langle v\rangle^{\alpha-n+1}}{|v|^{n-1+\alpha}}\,dv\right) =: \frac{\langle u\rangle}{|u|^{n-1-\alpha}}\left(J_\delta(u)+c_\varepsilon\right)\end{aligned} \tag{3.15}$$

with

$$c_\epsilon = 2^{1-n}\int_{|v|>\delta}\frac{\langle v\rangle^{n-1-\alpha}-1}{|v|^{n-1-\alpha}\langle v\rangle^{n-1-\alpha}}\,dv.$$

Passing to polar coordinates, we transform c_ϵ to $c_\epsilon = 2^{\alpha-(n-1)}c(\epsilon)$ with $c(\epsilon)$ given by (3.13). It remains to transform the term J_δ in (3.15). To this end, we make the same rotation change of variables as in the proof of the remark after Lemma 3.2 and get

$$J_\delta(u) = \frac{2^{1-n}}{\gamma}|u|^{n-1-\alpha}\int_{|y|>\frac{\delta}{|u|}}\frac{|y-e_1|^{\alpha-n+1}-1}{|y|^{n-1+\alpha}}\,dy.$$

Making the inversion change $y=\frac{t}{|t|^2}$, $dy=\frac{dt}{|t|^{2(n-1)}}$, and using the relation $\left|\frac{y}{|y|^2}-e_1\right| = \frac{|y-e_1|}{|y|}$, we obtain

$$J_\delta(u) = \frac{2^{1-n}}{\gamma|u|^\alpha}\int_{|y|<\frac{|u|}{\delta}}\left(|y-e_1|^{\alpha-n+1}-|y|^{\alpha-n+1}\right)\,dy,$$

which transforms (3.15) into (3.12)-(3.14). □

Lemma 3.4. *Let $0<\Re\alpha<2$. The kernel $\mathcal{K}_\epsilon(x\cdot\sigma)$ satisfies conditions i)-iii) of Theorem 3.1 and is therefore an identity approximation kernel.*

Proof. To check condition ii) of Theorem 3.1, we have, according to (3.14), to estimate the integral

$$\int_{S^{n-1}} |\mathcal{K}_\varepsilon(x\cdot\sigma)|\,d\sigma = \frac{2^{1-n}}{\delta^{n-1}}\int_{S^{n-1}} \langle u\rangle^{n-1-\alpha}\left|\mathcal{K}_\varepsilon\left(\frac{|u|}{\delta}\right)\right|\,d\sigma$$

where $\delta = \frac{\varepsilon}{\sqrt{4-\varepsilon^2}}$. We transform this integral to one over R^{n-1} by means of the change of variables (3.5). Using relation (2.19) for the stereographical projection, we arrive at

$$\int_{S^{n-1}}|\mathcal{K}_\varepsilon(x\cdot\sigma)|\,d\sigma = \frac{1}{\delta^{n-1}}\int_{R^{n-1}}\left|\mathcal{K}_{1,\alpha}\left(\frac{|x|}{\delta}\right)\right|\frac{du}{\langle u\rangle^{n-1+\alpha}} \le \int_{R^{n-1}}|\mathcal{K}_{1,\alpha}(|u|)|\,du<\infty$$

by the known fact that the kernel $\mathcal{K}_{1,\alpha}(|u|)$ is integrable as shown in [Sa1], Theorem 1 or [SKM], Lemma 26.4.

To check i), we proceed similarly:

$$\lim_{\varepsilon\to 0}\int_{S^{n-1}}\mathcal{K}_\varepsilon(x\cdot\sigma)\,d\sigma = \lim_{\delta\to 0}\int_{R^{n-1}}|\mathcal{K}_{1,\alpha}(|u|)\,\frac{du}{(1+\delta^2|u|^2)^{\frac{n-1+\alpha}{2}}} = 1$$

because $\mathcal{K}_{1,\alpha}(|u|)|$ is an averaging kernel; see [Sa1], formula (2.8) or [SKM], formula (26.42). Finally, condition iii) is verified in a similar fashion:

$$\lim_{\varepsilon\to 0}\int_{x\cdot\sigma<1-t_0}|\mathcal{K}_\varepsilon(x\cdot\sigma)|\,d\sigma = \lim_{\delta\to 0}\int\limits_{|u|>\frac{1}{\delta}\sqrt{\frac{t_0}{2-t_0}}}|\mathcal{K}_{1,\alpha}(|u|)|\,\frac{du}{(1+\delta^2|u|^2)^{\frac{n-1+\alpha}{2}}} = 0.$$

□

Theorem 3.5. *Let $0<\Re\alpha<2$. The operator*

$$T^\alpha = c_\alpha I + D^\alpha = c_\alpha I + \lim_{\substack{\epsilon\to 0\\(L_p)}} D^\alpha_\epsilon \tag{3.16}$$

with the constant c_α defined by (2.16) is a left inverse to $\mathcal{K}^\alpha$ in $L_p(S^{n-1})$, $1\le p<\infty$ and in $C(S^{n-1})$.

Proof. From (3.2) and (3.12) we have

$$(T^\alpha K^\alpha\varphi)(x) = c_\alpha(K^\alpha\varphi)(x) + \lim_{\substack{\varepsilon\to 0\\(L_p)}}\int_{S^{n-1}}\left[\frac{c(\varepsilon)}{|x-\sigma|^{n-1-\alpha}} + K_\varepsilon(x\cdot\sigma)\right]\varphi(\sigma)\,d\sigma,$$

or

$$(T^\alpha K^\alpha\varphi)(x) = \left[c_\alpha + \gamma_{n-1}(\alpha)\lim_{\varepsilon\to 0}c(\varepsilon)\right](K^\alpha\varphi)(x) + \lim_{\substack{\varepsilon\to 0\\(L_p)}}\int_{S^{n-1}}K_\varepsilon(x\cdot\sigma)\varphi(\sigma)\,d\sigma\,. \tag{3.17}$$

Evidently,

$$\begin{aligned}\gamma_{n-1}(\alpha)\lim_{\varepsilon\to 0} c(\varepsilon) &= \frac{2^{-\alpha}|S^{n-2}|}{\gamma_{n-1}(\alpha)}\int_0^\infty \frac{1-(1+r^2)^{\frac{\alpha+1-n}{2}}}{r^{1+\alpha}}\,dr \\ &= \frac{\Gamma\left(\frac{n-1+\alpha}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(-\frac{\alpha}{2}\right)}\int_0^\infty \frac{1-(1+t)^{\frac{\alpha+1-n}{2}}}{t^{1+\frac{\alpha}{2}}}\,dt\ .\end{aligned}$$

We make use of the formula

$$\int_0^\infty \frac{1-(1+t)^{-\mu}}{t^{1+\beta}}dt = \frac{\Gamma(\mu+\beta)\Gamma(1-\beta)}{\beta\Gamma(\mu)}\ , \quad 0<\Re\beta<1,\ \Re(\mu+\beta)>0,$$

which is easily obtained by integration by parts, and obtain

$$\gamma_{n-1}(\alpha)\lim_{\epsilon\to 0} c_\epsilon = \frac{2\Gamma(1-\frac{\alpha}{2})\Gamma(\frac{n-1+\alpha}{2})}{\alpha\Gamma(-\frac{\alpha}{2})\Gamma(\frac{n-1-\alpha}{2})} = -c_\alpha\ .$$

Therefore, by (3.17),

$$(T^\alpha K^\alpha\varphi)(x) = \lim_{\substack{\epsilon\to 0\\ (L_p)}}\int_{S^{n-1}}\mathcal{K}_\epsilon(x\cdot\sigma)\varphi(\sigma)d\sigma.$$

It remains to make use of Lemma 3.4. □

Remark. Since the spherical potential operator K^α can be transformed to a similar operator over R^{n-1}, see (2.20), we could try to use the known inversion results for the spatial case (see [Sa1], Theorem 2, or [SKM], Theorem 26.3). However, this way proves to be inappropriate because the natural truncation of the hypersingular integral in R^{n-1}, by the ball of the radius ϵ, is transformed by the stereographic projection, used in (2.20), into an artificial truncation on the sphere, which, in addition, depends on the pole of the projection. The spherical hypersingular integral truncated in such an induced way, proves to be non-commuting with rotations. Above we used the direct truncation, which is rotation invariant.

4. Inversion of the spherical Riesz potential in the case $\Re\alpha > 2$

To lower the order of the potential operator, it is natural to use the Beltrami-Laplace operator

$$\delta f = \mathcal{R}\Delta P f, \quad f = f(\sigma), \quad \sigma\in S^{n-1},$$

where $Pf = f\left(\frac{x}{|x|}\right)$ is the operator of the continuation of a function $f(\sigma)$ on S^{n-1} to a homogeneous function in R^n and $\mathcal{R}$ is the operator of restriction to S^{n-1}. In what follows the Beltrami-Laplace operator for a function $f(x)$ in the range of the operator K^α is treated in the distributional sense:

$$(\delta f,\psi) \;=\; (f,\delta\psi)\;, \quad \psi\in C^\infty(S^{n-1}). \tag{4.1}$$

Theorem 4.1. *Let* $f=K^\alpha\varphi,\ \varphi\in L_p(S^{n-1}),\ 1\le p<\infty,$ *where* $\Re\alpha>0,\ \Re\alpha\ne 2,4,6,\ldots$ *and* $\alpha\ne n-1+2k,\ k=0,1,2,3,\ldots$ *Then*

$$\varphi(x) \;=\; c_\beta g(x) \;+\; D^\beta g(x) \tag{4.2}$$

where $\beta=\alpha-2N,\ N=\left[\frac{\Re\alpha}{2}\right]$, *so that* $0<\beta<2,\ c_\beta$ *being the constant defined by (2.16) and* g *being the function*

$$g(x)=\mathcal{P}_N(-\delta)f(x), \tag{4.3}$$

where $\mathcal{P}_N(t)$ *is the polynomial*

$$\mathcal{P}_N(t) \;=\; (t-t_1)\cdots(t-t_N) \tag{4.4}$$

with the roots

$$t_j \;=\; \left(j+\frac{\beta+n-3}{2}\right)\left(j+\frac{\beta+1-n}{2}\right)\;. \tag{4.5}$$

The inversion (4.2) is also valid for $\alpha=2,4,6,\ldots.$ *In this case* $\varphi=g(x)=\mathcal{P}_{\frac{\alpha}{2}}(-\delta)f$.

Proof. We first construct the inverse operator on nice functions $\varphi\in C^\infty(S^{n-1})$. By the property $\Gamma(x+1)=x\Gamma(x)$ of the function $\Gamma(x)$, the spherical multiplier

$$\lambda_m(\alpha) \;=\; \frac{\Gamma\left(m+\frac{n-1-\alpha}{2}\right)}{\Gamma\left(m+\frac{n-1+\alpha}{2}\right)}$$

of the operator K^α is reduced to $\lambda_m(\beta),\ \beta=\alpha-2N$:

$$\lambda_m(\alpha) \;=\; \frac{\lambda_m(\beta)}{\left(m+\frac{n-1+\beta}{2}\right)_N\left(m-N+\frac{n-1-\beta}{2}\right)_N}$$

where $(a)_N$ stands for the Pochhammer symbol.

Hence, the operator inverse to K^α must have the multiplier

$$\frac{1}{\lambda_m(\alpha)}=\frac{[m^2+m(n-2)+\gamma_+(\gamma_--1)]\cdots[m^2+m(n-2)+(\gamma_++N-1)(\gamma_--N)]}{\lambda_m(\beta)}$$

where $\gamma_\pm=\frac{n-1\pm\beta}{2}$. In the numerator we have a polynomial in $t_m=m^2+m(n-2)$. The latter is the multiplier of the Beltrami-Laplace operator $-\delta$. Therefore,

$$\frac{1}{\lambda_m(\alpha)} \;=\; \frac{\mathcal{P}_N(t_m)}{\lambda_m(\beta)}\;, \tag{4.6}$$

where

$$\mathcal{P}_N(t) = \prod_{j=1}^{N} \left[t + (\gamma_+ + j - 1)(\gamma_- - j) \right],$$

which coincides with (4.4)-(4.5).

Thus, the inversion formula (4.2) is proved in the case of functions $\varphi \in C^\infty(S^{n-1})$. To extend it to the case where $\varphi \in L_p(S^{n-1})$, we observe that $\mathcal{P}_N(-\delta)f$ is defined in the distributional sense (4.1) on functions $f \ := K^\alpha \varphi$ with $\varphi \in L_p$ if $2N < \alpha$. Moreover,

$$\mathcal{P}_N(-\delta)f \ = \ K^{\alpha - 2N}\varphi \ , \tag{4.7}$$

since $(\mathcal{P}_N(-\delta)f, \psi) \ = \ (f, K^\alpha \mathcal{P}_N(-\delta)\psi) \ = \ (f, K^{\alpha-2N}\psi)$ for $\psi \in C^\infty(S^{n-1})$. Thus, it suffices to invert the right-hand side of (4.7) by means of Theorem 3.5. □

We mention the paper [Ru] in which there was developed a certain technique, based on the Cavaliery type principle (4.24), to organize finite differences of order ℓ with respect to the "shift" $M_f(x', t)$. This led to a construction of the operator inverse to the Riesz potential operator K^α for all $0 < \alpha < \infty$, although in terms of rather complicated constructions.

Finally, as an open question we note the development of the method of approximative inverse operators for spherical potential operators. In application to spatial potential operators this method was developed in a series of papers, see [ZN], [NS1], [NS2], [Sa3], [Sa4]. This idea is especially attractive in the case of large values of $\Re\alpha$. We hope to shed light on this question in the future.

Acknowledgements

Partially supported by Centro de Matemática Aplicada at Instituto Superior Técnico, Lisbon, and by Russian Funds of Fundamental Investigations, Grant 00-01-00046a.

References

[BBP] Berens, H., Butzer, P.L., Pawelke, S., Limitierungsverfahren von Reihen mehrdimensionaler Kugelfunktionen und deren Saturationsverhalten, *Publs. Res. Inst. Math. Sci., Ser. A.* **4** (1968), 201-268.

[GR] Gradshtein, I.S., Ryzhik, I.M., *Tables of Integrals, Sums, Series and Products*, 5th Edition, Academic Press, Inc. 1994.

[Mi] Mikhlin, S.G., *Multi-dimensional Singular Integrals and Integral Equations (Russian)*, Moscow, Fizmatgiz 1962. (English transl.: Pergamon Press, Oxford-New York-Paris , 1965)

[Mu] Müller, C., *Spherical Harmonics*, Lect. Notes in Math, Springer, Berlin 1966.

[Ne] Neri, U., *Singular Integrals*, Lect. Notes in Math, Springer, Berlin 1971.

[NS1] Nogin, V.A., Samko, S.G., Method of approximating inverse operators and its applications to inversion of potential type integral operators, *Integr. Transforms and Special Funct.* **8** (1999), 89-104.

[NS2] Nogin, V.A., Samko, S.G., Some applications of potentials and approximative inverse operators in multi-dimensional fractional calculus, *Fract. Calculus and Appl. Anal.* **2** (1999), 205-228.

[PS1] Pavlov, P.M., Samko, S.G., A description of the space $L_p^\alpha(S_{n-1})$ in terms of spherical hypersingular integrals (Russian) , *Dokl. Akad. Nauk SSSR, ser. Matem.* **276** (1984), 546-550 (*Transl. in Soviet Math. Dokl.*, 29 (1984), 549-553).

[PS2] Pavlov, P.M., Samko, S.G., Inversion of Riesz potentials on a sphere and a characterization of the space $L_p^\alpha(S_{n-1})$ in terms of hypersingular integrals (Russian), *Deposited in VINITI, Moscow* **3800-85** (1985), 1-17 .

[Ru] Rubin, B.S., The inversion of fractional integrals on a sphere, *Israel. J. Math.* **79** (1992), 47-81.

[Sa1] Samko, S.G., Spaces of Riesz potentials (Russian), *Izv. Akad. Nauk SSSR, ser. Matem.* **40** (1976), 1143-1172 (*Transl. in Math. USSR Izvestija* **10** (1976), no 5, 1089-1117).

[Sa2] Samko, S.G., Generalized Riesz potentials and hypersingular integrals with homogeneous characteristics; their symbols and inversion (Russian), *Trudy Mat. Inst. Steklov* **156** (1980), 157-222 (*Transl. in Proc. Steklov Inst. Mat.* (1983), Issue 2, 173-243).

[Sa3] Samko, S.G., A new approach to the inversion of the of the Riesz potential operator, *Fract. Calculus and Appl. Anal.* **1** (1998), 225-245.

[Sa4] Samko, S.G., Some remarks to author's paper "A new approach to the inversion of the Riesz potential operator", *Fract. Calculus and Appl. Anal.* **2** (1999), 63-66.

[Sa] Samko, S.G., *Hypersingular Integrals and Their Applications*, Taylor & Francis, Series "Analytic Methods and Special Functions", v.5, 2002.

[SKM] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integrals and Derivatives. Theory and Applications*, Gordon & Breach Sci. Publ. 1993.

[SW] Stein, E., Weiss, G., *Introduction to Fourier Analysis on Euclidean Space*, Princeton Univ. Press 1971.

[ZN] Zavolzhenskii, M.M., Nogin, V.A., Approximating approach to inversion of the generalized Riesz potentials, *Dokl. Akad. Nauk Rossii, ser. matem.* **324** (1992), 738-741.

Universidade do Algarve
Unidade de Ciencias Exactas
Campus de Gambelas
Faro 8000
Portugal

1991 Mathematics Subject Classification. Primary 47G10, Secondary 45P05, 26A33

Received 10 February 2001